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Quantum strings in $AdS_5 \times S^5$: strong-coupling corrections to dimension of Konishi operator

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Abstract: We consider leading strong coupling corrections to the energy of the lightest massive string modes in $AdS_5 \times S^5$, which should be dual to members of the Konishi operator multiplet in $\mathcal{N} = 4$ SYM theory. This determines the general structure of the strong-coupling expansion of the anomalous dimension of the Konishi operator. We use 1-loop results for several semiclassical string states to extract information about the leading coefficients in this expansion. Our prediction is $\Delta = 2\lambda^{1/4} + b_0 + b_1\lambda^{-1/4} + b_3\lambda^{-3/4} + ...$, where b_0 and b_1 are rational while b_3 is transcendental (containing $\zeta(3)$). Explicitly, we argue that $b_0 = \Delta_0 - 4$ (where Δ_0 is the canonical dimension of the corresponding gauge-theory operator in the Konishi multiplet) and $b_1 = 1$. Our conclusions are sensitive to few assumptions, implied by a correspondence with flat-space expressions, on how to translate semiclassical quantization results into predictions for the exact quantum string spectrum.

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Contents

1 Introduction

The canonical example of the AdS/CFT duality [1–3] implies the equivalence between the spectrum of the planar $\mathcal{N} = 4$ SYM theory and the spectrum of free quantum string in $AdS_5 \times S^5$space. The spectrum of the gauge theory can be described either as a list of possible energies of SYM states on $R \times S^3$ (as functions of various quantum numbers) or as a list of dimensions Δ of conformal primary operators on $R^{1,3}$ (determined by diagonalisation of anomalous dimension matrix for single-trace gauge-invariant operators). Similarly, the string spectrum is given by the AdS_5 energies E of string states on a cylinder $R \times S^1$ (found using,e.g., a light-cone gauge approach) or is found from the marginality condition for the corresponding string vertex operators on a plane $R^{1,1}$ (by diagonalizing of the 2-d anomalous dimension matrix).

Below we will be interested in the strong coupling expansion of dimensions of gauge theory operators or inverse string tension expansion of energies of the corresponding quantum string states.

To set up the notation we will be using below, we will label representations of the bosonic subgroup $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ of the symmetry group $PSU(2,2|4)$ by the Young tableaux labels

$$\hat{\mathrm{C}} = (E, S_1, S_2; J_1, J_2, J_3) \equiv (E, \mathrm{C}) \ . \tag{1.1}$$

Here each charge of the highest-weight state corresponds to six SO(2) subgroups with $\mathrm{C} = (S_1, S_2; J_1, J_2, J_3)$ being the spins. These are related to the often used SU(2) $\times$ SU(2) labels (s_L, s_R) for SO(4) and the SU(4) Dynkin labels $[p_1, q, p_2]$ as: $s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$, and $p_{1,2} = J_2 \mp J_3, \ q = J_1 - J_2$, i.e.

$$\mathrm{C} = [J_2 - J_3, J_1 - J_2, J_2 + J_3]_{\left(\frac{S_1+S_2}{2}, \frac{S_1-S_2}{2}\right)} \ . \tag{1.2}$$

Then the equivalence of the gauge and string theory spectra can be expressed as

$$\Delta(\lambda, \mathrm{C}) = E(\sqrt{\lambda}, \mathrm{C}) \,, \tag{1.3}$$

where $\Delta = E_{\rm gauge}, \quad E = E_{\rm string}$, λ is the 't Hooft coupling and $\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$ is the $AdS_5 \times S^5$string tension.[1]

In the weak-coupling ($\lambda \ll 1$) expansion represented by the perturbative gauge theory

$$\Delta = \Delta_0 + \gamma(\lambda, \mathrm{C}) \,, \qquad\qquad \gamma = k_1 \lambda + k_2 \lambda^2 + \ldots \,, \tag{1.4}$$

where Δ_0 is the canonical dimension of the corresponding operator. γ is an eigenvalue of the 4-d anomalous dimension matrix. Only the operators with the same Δ_0 can mix, so Δ_0 may be called a "level" of gauge-theory states. Δ_0 may change for states within the same supermultiplet, as dictated by the commutation relations of $PSU(2,2|4)$, while γ should be the same.

In the strong-coupling ($\lambda \gg 1$) expansion represented by the perturbative (inverse string tension) expansion in the string-theory sigma model, one may expect that for large λ (or large radius $R \gg \sqrt{\alpha'}$ of $AdS_5 \times S^5$space) massive quantum string states or "short" strings with fixed charges C probe a near-flat region of $AdS_5 \times S^5$and thus their energies may be found by a near-flat-space expansion. Then one may expect to find

$$E(\sqrt{\lambda}, \mathrm{C}) = 2\sqrt{n-1}\sqrt[4]{\lambda} \ + \ \sum_{k=0}^{\infty} \frac{b_k}{(\sqrt[4]{\lambda})^k} \ . \tag{1.5}$$

Here the leading term [2] is the analog of the flat-space string mass term (originating from $\alpha' E^2 = 4(n-1)$) with n being the flat-space string level.[2] The structure of corrections may be, in principle, determined from diagonalization of the 2-d anomalous dimension matrix for the corresponding string vertex operators (see [4, 5] and below) having the same canonical 2-d dimension $2n$, i.e. representing states from the same string level n.[3] The 2-d

[1] Here we suppressed any potential dependence of Δ or E on various other "hidden" charges that specify the gauge theory operators and the quantum string states.

[2] We shall use the NSR definition of string level, with $n = 1$ corresponding to massless level and $n = 2$ to the first excited level. For some earlier discussions of energies of quantum string states in $AdS_5 \times S^5$see also [4, 5, 7–11]. In particular, an expansion of the form (1.5) appeared in the fermionic model for the $su(1|1)$ sector in [9].

[3] The 2-d operators that may mix must have the same n. This follows, e.g., from momentum conservation when computing the 2-point functions in world-sheet perturbation theory.

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anomalous dimensions are given by a *regular* series expansion in $\alpha' = \frac{1}{\sqrt{\lambda}}$, while $\frac{1}{(\sqrt[4]{\lambda})^k}$ appear as a result of solving quadratic-type equations for E following from the marginality condition.[4]

States belonging to the same supermultiplet must have the same n but may have different values of b_0, which should differ by the same amount as the canonical dimension Δ_0 in (1.4).

It is useful to split the sum in (1.5) into the "odd" ($\frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_3}{(\sqrt[4]{\lambda})^3} + ...$) and "even" ($b_0 + \frac{b_2}{(\sqrt[4]{\lambda})^2} + ...$) power parts as these appear to have different origin within the semiclassical expansion we shall use to determine the strong-coupling coefficients b_k. Then one can also rewrite (1.5) as

$$E = E^{(\mathrm{an})} + E^{(\mathrm{nan})}\,, \tag{1.6}$$

$$E^{(\mathrm{an})} = \sqrt{\sqrt{\lambda}}\left[2\sqrt{n-1} + \frac{b_1}{\sqrt{\lambda}} + \frac{b_3}{(\sqrt{\lambda})^2} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^3}\right)\right], \tag{1.7}$$

$$E^{(\mathrm{nan})} = b_0 + \frac{b_2}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)\ . \tag{1.8}$$

As we shall see, in the semiclassical approach to the energy of strings with small values of spins the "analytic" part $E^{(\mathrm{an})}$ is the one that originates from the classical string energy and also from a "regular" part of semiclassical corrections (e.g., determined by even powers of masses of string fluctuations) while the "non-analytic" part $E^{(\mathrm{nan})}$ has its origin, from semiclassical standpoint, in certain special IR parts of quantum corrections (which are due to zero or "light" modes that become massless in the "small-spin" string limit).[5]

The weak-coupling expansion (1.4) given by the planar 4-d perturbation theory should have finite radius of convergence and thus should define $\Delta(\lambda, \mathrm{C})$ for all values of λ. Expanding the resulting function at large λ one should then reproduce the strong-coupling expansion (1.5) as predicted by the string theory. Once λ is increased so that the anomalous dimension γ becomes of the same order as Δ_0, the latter looses its "invariant" meaning. An interesting question is how the value of Δ_0 is encoded in the strong-coupling expansion coefficients in (1.5). And vice versa, the meaning of the string level n in (1.5) in the weak-coupling gauge theory expansion (1.4) is also unclear a priori.

Our aim below will be to clarify the general structure of the strong coupling expansion (1.5), (1.6) on examples of string states at the first excited level $n = 2$ which are dual to members of the Konishi operator multiplet in gauge theory. We shall use 1-loop string results for several semiclassical string states to extract information about the two leading coefficients b_0 and b_1 in (1.7), (1.8). Our results for the two subleading coefficients in the dimension of the members of the Konishi operator multiplet (with $\Delta_0 = 2, ..., 10$) may be summarized as follows:

$$n = 2: \qquad b_0 = \Delta_0 - 4\,, \qquad b_1 = 1\ . \tag{1.9}$$

[4]In particular, there cannot be any $\log \lambda$ terms such as those that appear in the strong-coupling expansion of the anomalous dimensions computed using asymptotic Bethe ansatz equations [6].

[5]This distinction into "analytic" and "non-analytic" terms in the 1-loop energy in the small-spin limit, which has an IR origin, should not be confused with the one in [12] which appeared in the large-spin limit and had an UV origin.

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We shall also conjecture that

$$b_2 = 0 \ . \tag{1.10}$$

The rest of the paper is organized as follows. We shall start in section 2 with general remarks on the structure of the strong-coupling expansion (1.5) explaining how it follows from solving the marginality conditions for the corresponding string vertex operators. We shall consider constraints on the 1-loop 2-d anomalous dimension implied by the structure of the Konishi supermultiplet and identify, from this standpoint, the origin of the two components $E^{(\rm an)}$ and $E^{(\rm nan)}$. We shall also discuss form of the 2-d anomalous dimensions of the corresponding composite operators as determined by the bosonic part of the $AdS_5 \times S^5$string sigma model

To systematically include the effects of fermions in section 3 we propose to use a different strategy: start with semiclassical spinning string solutions, compute 1-loop corrections to their energies and then attempt to interpolate to small values of spins corresponding to states at the first excited string level. We end up with what appears to be a consistent picture with different types of spinning string states finding their counterparts among the states in the Konishi multiplet table and predicting the same universal expression for the corresponding anomalous dimension. Our results are summarised in section 4.

2 General structure of strong-coupling expansion

There are few guiding principles that one may try to use to understand the interpolation of dimensions of composite operators from weak to strong coupling. First, one may expect the validity of a "non-intersection principle" [4]: there should be no level crossings for states with the same quantum numbers as λ changes from weak to strong coupling. That would suggest that (for fixed values of charges) the states with smaller values of the gauge-theory "level" Δ_0 and thus smaller dimension at weak coupling should correspond to states with smaller energy also at strong coupling. The singlet Konishi scalar operator with $\Delta_0 = 2$ which has lowest dimension at weak coupling should correspond to a string state on the first excited level $n = 2$. In fact, the analysis based on symmetries and near flat space expansion suggests that the states of the Konishi supermultiplet [14] should belong [13, 15] to the set of the superstring states at the level $n = 2$.[6]

Second, since gauge-theory states belonging to the same supermultiplet should have the same anomalous dimension (while their Δ_0's may differ by (half)integer values as they are related by application of supersymmetry generators) the equality (1.3) suggests, in view of (1.5), that for the corresponding string states

$$E = 2\sqrt{n-1}\ \sqrt[4]{\lambda}\ + \Delta_0 + \mathrm{b}_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt[4]{\lambda})^3}\right) , \qquad b_0 = \Delta_0 + \mathrm{b}_0 \, , \tag{2.1}$$

[6] More precisely, the Konishi supermultiplet should be the $J = 0$ Kaluza-Klein "floor" of the whole set of states at the first excited level given by $\sum_{J=0}^{\infty}[0, J, 0]\times$ [Konishi multiplet] [13]. Adding extra S^5 orbital momentum J increases canonical dimension of the gauge-theory operator but does not increase the level of the dual string state.

where the coefficients $n, \mathrm{b}_0, b_1, b_2, ...$ appearing in the strong-coupling expansion of the anomalous dimension γ should be universal, i.e. should be the same for all the states in a supermultiplet.[7]

In contrast to the weak-coupling region (1.4) where members of the same supermultiplet may have very different dimensions as Δ_0 may jump from state to state, at strong coupling (1.5) all dimensions of states from the same level are approximately equal, differing only in the subleading terms controlled again by Δ_0 part of b_0. While in flat space all string states at a given level have the same mass or rest-frame energy, switching on the curvature removes this degeneracy.[8]

The main problem is how to compute the quantum dimensions of the corresponding vertex operators and thus determine the coefficients in the expansion (1.5) of E. As we shall discuss in more detail in section 2.2 below, one expects the leading terms in the (eigenvalue of the) 2-d dimension of the vertex operator representing string states with charges $\hat{\mathrm{C}} = (E, \mathrm{C})$ to be a generalization of the flat-space marginality condition $2 = 2n - \frac{\alpha'}{2}(E^2 - p_i^2)$. In $AdS_5 \times S^5$the term E^2 is replaced by a certain quadratic combination of the relevant charges. For example, for a state carrying a spin J we may a priori expect

$$2 = 2n - \frac{1}{2\sqrt{\lambda}}\Big[E(E + a_1) + a_2 EJ + a_3 J(J + a_4) + a_5\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) . \qquad (2.2)$$

The expansion of the 2-d anomalous dimension goes in integer powers of the inverse string tension, i.e. contains only $\frac{1}{(\sqrt{\lambda})^k}$-terms.[9]

The structure of (2.2) is implied also by the space-time interpretation of the 2-d anomalous dimension operator as a differential operator acting on the corresponding tensor coefficients Ψ of a basis of vertex operators. For a flat-space state with mass $m_0^2 = \frac{4(n-1)}{\alpha'}$, one expects to find in curved background

$$\left[2 - 2n + \frac{\alpha'}{2}\nabla^2 + \alpha'(c_1 \mathrm{R} + c_2 \mathrm{F}_5 \mathrm{F}_5) + \mathcal{O}(\alpha'^2)\right]\Psi = 0\,, \qquad (2.3)$$

where R stands for the curvature tensor and F_5 stands for the 5-form field strength (the α' term may contain several tensor structures, cf. [10]). Due to the large amount of supersymmetry, higher α'^k corrections to the "mass matrix" are expected (on the basis of the NS-NS sector experience) to start at relatively late order.[10]

[7] This expression is indeed consistent with the non-intersection principle: the states with the same charges (at the same string level or in the same supermultiplet) that had smaller dimension (i.e. smaller Δ_0) at weak coupling will have smaller dimension also at strong coupling.

[8] Note also that, similarly to weakly-coupled gauge theory where the operators can be constructed in terms of the free-theory fields, at strong coupling or in the near-flat-space expansion one may label string states by the oscillator numbers of the flat-space superstring description.

[9] To compute $\frac{1}{\sqrt{\lambda}}$ corrections to the canonical 2-d dimension $2n$-term one is supposed to choose a basis of composite operators consistent with symmetries, compute the anomalous dimension matrix using string sigma model perturbation theory and then diagonalize this matrix. The resulting eigen-operators will be given by linear combinations of operators from the basis (with coefficients that may depend on $\frac{1}{\sqrt{\lambda}}$). These will be conformal primaries that have definite dimensions and define string vertex operators (that can be used also to compute correlation functions and thus string scattering amplitudes, etc.).

[10] This suggests that non-trivial corrections in (2.2) should be postponed at least till order $\frac{1}{(\sqrt{\lambda})^3}$.

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The expression (2.1) for $E(\sqrt{\lambda}, \mathrm{C})$ then follows from (2.2) by solving it perturbatively in $\frac{1}{\sqrt{\lambda}}$. For the lowest level (supergravity or BPS) states with $n = 1$ each $\frac{1}{(\sqrt{\lambda})^n}$ term in (2.2) should vanish separately so that E should not depend on $\sqrt{\lambda}$. For massive string states with $n > 1$ and for fixed charges $\mathrm{C} = (J, ...)$ we get from (2.2)

$$E^2 - 2q_0 E + q_1 = 4(n-1)\sqrt{\lambda} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) . \tag{2.4}$$

Solving this quadratic equation produces terms with powers of square root of string tension, i.e. leads to (2.1) with

$$b_0 = q_0 \,, \qquad b_1 = \frac{q_0^2 - q_1}{4\sqrt{n-1}} \,. \tag{2.5}$$

i.e. reproduces the structure of the strong-coupling expansion anticipated in (1.5). It is then clear that any effective $\frac{1}{\sqrt{\lambda}}$ corrections to the q_1 and q_2 terms coming from $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$ term in (2.4) will be subleading compared to the three leading terms in (2.5). Thus b_0 and b_1 are determined by the 1-loop correction to the 2-d anomalous dimension in (2.2).

Let us note also that the b_2-term in (1.8) may appear only from the *2-loop* $\frac{1}{(\sqrt{\lambda})^2}E$ term in (2.2) (which effectively shifts the coefficient $q_0 \to q_0 + \frac{c}{\sqrt{\lambda}}$ in (2.4)). As was mentioned above, it seems likely that such terms should not appear due to supersymmetry so we conjecture (1.10) that $b_2 = 0$.

2.1 Supersymmetry constraints: Konishi supermultiplet

In the case of the states from the first excited string level that are expected to correspond to states of the Konishi multiplet we find from (2.1), (2.4)

$$E^2 - 2(\Delta_0 + \mathrm{b}_0)E + (\Delta_0 + \mathrm{b}_0)^2 - 4b_1 = 4\sqrt{\lambda} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) , \tag{2.6}$$

$$E = 2\sqrt[4]{\lambda} + \Delta_0 + \mathrm{b}_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt[4]{\lambda})^2}\right) . \tag{2.7}$$

Here b_0 and b_1 should be universal within the multiplet while Δ_0 may change from 2 to 10 in steps of 1/2.

To further clarify the origin of (2.6), (2.7) let us study to which degree the strong-coupling expansion of E is controlled by $PSU(2,2|4)$ symmetry that determines the structure of the Konishi multiplet [13, 14] listed in table 1 (we borrow this table from [13]). For every state in the table 1 there should exist a vertex operator at the level $n = 2$. For each operator we should get the same value for the 4-d anomalous dimension $\gamma = \Delta - \Delta_0 = E - \Delta_0$ (which is the only quantity undetermined by the representation theory) by solving the 2-d marginality condition.

As discussed below in section 2.2, the 1-loop correction to the 2-d anomalous dimension in (2.2) may be at most quadratic in the charges $\hat{\mathrm{C}}_A = (E, S_1, S_2; J_1, J_2, J_3)$ (the corresponding 1-loop Feynman diagrams involve at most two of the fields of the operator at a time). Then the general form of the 1-loop marginality condition will be (cf. (2.2) for $n = 2$)

$$2 = 4 - \frac{1}{2\sqrt{\lambda}}\Big(\sum_{A,B=1}^{6} u_{AB}\hat{\mathrm{C}}_A\hat{\mathrm{C}}_B + \sum_{A=1}^{6} v_{\ell A}\hat{\mathrm{C}}_A + h_\ell\Big) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) . \tag{2.8}$$

Here u, v, h are constant coefficients and we introduced dependence on the supersymmetry "level" $\ell = 0, 1, .., 16$ of the supermultiplet, with

$$\Delta_0 = 2 + \frac{1}{2}\ell . \tag{2.9}$$

One may argue that the coefficients u_{AB} should not depend on ℓ: the action of the supersymmetry generators only changes charges of a given state by a finite amount (e.g., $J_1 \to J_1 + \frac{1}{2}\ell$, etc) so that the terms quadratic in the charges do not acquire any ℓ-dependence.

Solving the condition (2.8) for E for all of the bosonic states in the Konishi multiplet whose charges are listed in table 1[11] and requiring that E jumps by $\frac{1}{2}(\ell_2 - \ell_1)$ when going from a supermultiplet level ℓ_2 to level ℓ_1 one can determine the coefficients in (2.8) and finally obtain the following expression for E in (2.7) with

$$\mathrm{b}_0 = -2 + \frac{1}{2}(h_2 - h_0 - 1) , \qquad b_1 = \frac{1}{16}(h_2 - h_0 - 1)^2 - \frac{1}{4}h_0 . \tag{2.10}$$

Here h_0 and h_2 are undetermined universal (i.e. ℓ-independent) constants. The marginality conditions for states at different supermultiplet levels $\ell = 2\Delta_0 - 4$ then follow from (2.7). These give expressions for the 2-d anomalous dimensions of the vertex operators obtained by acting with ℓ supersymmetry generators on the one corresponding to the "lowest" state in the supermultiplet.[12]

It is interesting to note that for the values of b_0 and b_1 in (1.9) we shall find below (i.e. $\mathrm{b}_0 = -4,\ b_1 = 1$) the relations (2.10) imply

$$h_0 = 0 , \qquad h_2 = -3 . \tag{2.11}$$

The value of $h_0 = 0$ in (2.8) appears indeed to be very natural for lowest-level state in the Konishi supermultiplet.

2.2 Structure of 2-d anomalous dimensions of vertex operators

To give an idea of how one could compute the 2-d anomalous dimension (and thus the values of b_0 and b_1 in (2.7)) from first principles let us review the structure of the corresponding

[11] It suffices to do this only for states up to (and including) those with $\Delta_0 = 6$ since the states with higher Δ_0 can be found by conjugation.

[12] Repeating similar analysis in the case of the short multiplet of BPS (supergravity) states starting with the $[0, J, 0]_{(0,0)}$ KK scalar state one finds that the analog of (2.2) is

$$2 = 2 - \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) - (J+\ell)(J+4-\ell)\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) ,$$

where $\ell = 1, 2, 3, 4$ corresponds to the "level" of the bosonic states in the supermultiplet.

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(bosonic) vertex operators following [4, 5]. The action of the $AdS_5 \times S^5$superstring sigma model [16] written in terms of the 6+6 embedding coordinates has the following structure

$$
\begin{aligned}
I &= \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big(- \partial N_a \bar{\partial} N^a + \partial n_k \bar{\partial} n_k + \text{fermions} \Big) \,, && (2.12) \\
N_a N^a &= N_+ N_+^* - N_x N_x^* - N_y N_y^* = 1, \qquad n_k n_k = n_x n_x^* + n_y n_y^* + n_z n_z^* = 1, && (2.13)
\end{aligned}
$$

where $N_+ = N_0 + iN_5,\ N_x = N_1 + iN_2,\ N_y = N_3 + iN_4,\ n_x = n_1 + in_2,\ n_x = n_3 + in_4,\ n_z = n_5 + in_6$. The fermions make this model UV finite. The aim is to construct marginal (1,1) vertex operators in terms of N_a, n_k and the fermions which correspond to the highest weight states of $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ representations.

For example, the vertex operator for dilaton-type massless level $n = 1$ (supergravity) scalar mode with SO(6) spin J should have the structure[13]

$$
V_J^{(0)} = (N_+)^{-E}\ (n_x)^J\ (-\partial N_a \bar{\partial} N^a + \partial n_k \bar{\partial} n_k + \text{fermions})\ . \tag{2.14}
$$

The corresponding marginality condition is (cf. (2.2))

$$
2 = 2 - \frac{1}{2\sqrt{\lambda}} \Big[E(E-4) - J(J+4) \Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2} \right) , \tag{2.15}
$$

so that to the 1-loop order $E = 4 + J$ and all higher-order corrections should vanish as this should be a BPS state.

In flat-space string theory a spin S state on the leading Regge trajectory is represented by (ignoring fermionic terms) $V_S = e^{-iEt} \big(\partial \mathrm{x}_x \bar{\partial} \mathrm{x}_x \big)^{\frac{S}{2}},\ \mathrm{x}_x = x_1 + ix_2$, with the marginality condition being $2 = S - \frac{1}{2}\alpha' E^2 = 0$, i.e. $E = \sqrt{\frac{2}{\alpha'}(S-2)}$. By analogy, in $AdS_5 \times S^5$case some candidate operators for states on the leading Regge trajectory are

$$
V_J = (N_+)^{-E} \big(\partial n_x \bar{\partial} n_x \big)^{\frac{J}{2}} + \dots , \qquad V_S = (N_+)^{-E} \big(\partial N_x \bar{\partial} N_x \big)^{\frac{S}{2}} + \dots , \tag{2.16}
$$

where dots stand for the fermionic terms and $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms resulting from diagonalization of the anomalous dimension operator. In general, ignoring the fermions, the operator $\big(\partial n_x \bar{\partial} n_x \big)^{\frac{J}{2}}$ in the SO(6) sigma model may mix with

$$
(n_x)^{2p+2q} (\partial n_x)^{\frac{J}{2} - 2p} (\bar{\partial} n_x)^{\frac{J}{2} - 2q} (\partial n_m \partial n_m)^p (\bar{\partial} n_k \partial n_k)^q \,, \tag{2.17}
$$

where $p, q = 0, ..., \frac{J}{4};\ m, k = 1, ..., 6$. The operator $(N_+)^{-E} \big(\partial N_x \bar{\partial} N_x \big)^{\frac{S}{2}}$ in the $\mathrm{SO}(2,4)$ sigma model may mix with

$$
(N_+)^{-E-p-q} N_x^{p+q} (\partial N_+)^p (\partial N_x)^{\frac{S}{2} - p} (\bar{\partial} N_+)^q (\bar{\partial} N_x)^{\frac{S}{2} - q} + O(\partial N_a \partial N^a \bar{\partial} N_b \bar{\partial} N^b) \,, \tag{2.18}
$$

where $p, q = 0, ..., \frac{S}{4};\ a, b = 0, 1, ...5$. The true vertex operators are eigenstates of the anomalous dimension matrix, i.e. particular linear combinations of the above structures.

[13] Recall that $N_+ = \cosh\rho\ e^{it}$ where t is AdS_5 global time coordinate and also $n_x \sim e^{i\varphi}$ where φ is an isometric angle of S^5.

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These could, in principle, be found by solving Lichnerowitz-operator type equations expressing marginality condition. In the case of the bosonic model $I = \frac{1}{4\pi\alpha'}\int d^2\sigma G_{mn}(x)\partial x^m \bar{\partial} x^n$ perturbed, e.g., by $V = \Psi_{m_1...m_J}(x)\partial x^{m_1}...\bar{\partial} x^{m_J}$ one could find the 2-d anomalous dimension by computing the renormalization of $\Psi_{m_1...m_J}$ and setting $\beta_\Psi = \hat{\gamma}\Psi + \mathcal{O}(\Psi^2)$=0. That would give (cf. [17])

$$\hat{\gamma}\Psi = \left[2 - J + \frac{1}{2}\alpha'\nabla^2 + \sum c_k \alpha'^k (R....)^n ... \nabla^p\right]\Psi = 0 \ . \tag{2.19}$$

Solving this equation for Ψ would amount to finding the eigen-states of $\hat{\gamma}$. However, the general form of $\hat{\gamma}$ for generic Ψ and curved background is not known even to the leading (1-loop) order in α'.[14] For that reason one apparently is to resort to "first-principles" computation for each specific model.

For example, the operators in the SO(6) model that are relevant for states on leading Regge trajectory (i.e. containing no terms with $\partial^k n$, $k > 1$) are

$$O_{\ell,s} = \Psi_{k_1...k_\ell m_1...m_{2s}} n_{k_1}...n_{k_\ell}\partial n_{m_1}\bar{\partial} n_{m_2}...\partial n_{m_{2s-1}}\bar{\partial} n_{m_{2s}} \ . \tag{2.20}$$

Their renormalization was studied in [5, 18–20]. The simplest case is $\Psi_{k_1...k_\ell} n_{k_1}...n_{k_\ell}$ with traceless $\Psi_{k_1...k_\ell}$ which is mapped by renormalization into itself and has the same 2-d anomalous dimension as its highest-weight representative $(n_x)^J$, i.e. $-\frac{1}{2\sqrt{\lambda}}J(J+4) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)$; it corresponds to a scalar spherical harmonic that solves the Laplace equation on S^5.

Similar results are found for SO(2, 4) model by replacing $(n_x)^J$ and $\partial n_k \bar{\partial} n_k$ with, respectively, $(N_+)^{-E}$ and $\partial N_a \bar{\partial} N^a$, and reversing the sign of the coupling, $\frac{1}{\sqrt{\lambda}} \to -\frac{1}{\sqrt{\lambda}}$. Then the dimension of $(n_x)^J \partial n_k \bar{\partial} n_k$, i.e. $-2 - \frac{1}{2\sqrt{\lambda}}J(J+4) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)$ translates into the dimension of $(N_+)^{-E}\partial N_a \bar{\partial} N^a$, i.e. $-2 + \frac{1}{2\sqrt{\lambda}}E(E-4) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)$, etc.

The number of $\partial n_k \bar{\partial} n_k$ factors in an operator like (2.20) never increases [19] and thus can be used as a "quantum number" to characterise the leading term in an eigen-operator. An example of a scalar operator carrying no spins is

$$V_r = (N_+)^{-E}\Big[(\partial n_k \bar{\partial} n_k)^r + ...\Big] \ , \tag{2.21}$$

for which the 1-loop and 2-loop terms in the 2-d dimension in bosonic $AdS_5 \times S^5$model are [4, 18–20]

$$\begin{aligned} \hat{\gamma}(V_r) = 2 - 2r &+ \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) + 2r(r-1)\Big] \\ &+ \frac{1}{(\sqrt{\lambda})^2}\left[\frac{2}{3}r(r-1)\left(r - \frac{7}{2}\right) + 4r\right] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^3}\right) \ . \end{aligned} \tag{2.22}$$

This operator corresponds to a scalar string state at level $n = r$, so the fermionic contributions should make the $r = 1$ state BPS, with $E = 4$ following from the $\hat{\gamma} = 0$

[14] Few exceptions are the WZW models (and models related to them by simple transformations like T-duality) and some plane-wave models.

condition. The $r = 2$ choice should correspond to a scalar state on the first excited string level. Eq. (2.22) implies then (cf. (2.4), (2.6)): $E(E-4) = 4\sqrt{\lambda} - 4 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$, so that $E = 2\sqrt[4]{\lambda} + 2 + \frac{0}{\sqrt[4]{\lambda}} + \mathcal{O}(\frac{1}{(\sqrt[4]{\lambda})^3})$. This result should not, however, be trusted as the fermions are expected to change the E-independent terms in the 1-loop anomalous dimension.

An example of another singlet scalar operator is $(N_+)^{-E}(\partial n_k \partial n_k \bar{\partial} n_m \bar{\partial} n_m)^q$ with $\hat{\gamma} = 2 - 4q + \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) + 16q\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)$, with $q = 1$ corresponding to a state on the first excited string level.

Going back to the operator in (2.16) for a string state with a spin J in S^5, we get

$$\hat{\gamma}(V_J) = 2 - J + \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) - \frac{1}{2}J(J+10)\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) . \tag{2.23}$$

The inclusion of the fermionic contributions may shift the coefficient of the term linear in J.

An example of a (bosonic) operator with two spins (J_1, J_2) in S^5 is [19] (cf. (2.13))

$$V_{J_1,J_2} = (N_+)^{-E} \sum_{u,v=0}^{J_2/2} c_{uv} n_y^{J_1-u-v} n_x^{u+v} (\partial n_y)^u (\partial n_x)^{\frac{J_2}{2}-u} (\bar{\partial} n_y)^v (\bar{\partial} n_x)^{\frac{J_2}{2}-v} , \tag{2.24}$$

where c_{uv} are constant coefficients. Ignoring the fermionic contributions, the highest and the lowest eigenvalues of the resulting 1-loop anomalous dimension matrix are [5]

$$\begin{aligned}
\hat{\gamma}_{\min} &= 2 - J_2 + \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) - J_1(J_1+4) - 2J_1J_2 - \frac{1}{2}J_2(J_2+10)\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right), \\
\hat{\gamma}_{\max} &= 2 - J_2 + \frac{1}{2\sqrt{\lambda}}\Big[E(E-4) - J_1(J_1+4) - J_2(J_2+6)\Big] + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) .
\end{aligned} \tag{2.25}$$

The fermionic contributions may again alter the coefficients of the terms linear in J_i and may be also produce a constant term like h_ℓ in (2.8).

Unfortunately, we do not know at present how to systematically incorporate the fermionic terms into the above vertex operators and thus how to compute the fermionic contributions to the 2-d anomalous dimensions starting with the $AdS_5 \times S^5$superstring action of [16].

One possible indirect approach towards determining these anomalous dimensions may be to reconstruct the quadratic term in the space-time effective action for the coefficient functions Ψ in, e.g., (2.20) and thus determine the leading terms in the equations (2.3), (2.19). This could be done, in principle, by reconstructing this effective action from the superstring flat-space S-matrix for massive string states using the NSR approach [10]. This approach, however, contains potential subtleties and we will not follow it here.[15]

Instead, below we will use the "semiclassical" approach to computation of energies of "short" string states that was initiated in [11]. It is based on the full $AdS_5 \times S^5$superstring

[15] One of the subtle issues (cf. [10]) is related to possible mixing of string states with different masses in 3-point amplitudes and the need to understand all such mixings in order to extract the "two massive — two massless" 4-point terms in the effective action.

action and thus incorporates the fermionic contributions but it requires certain assumptions of how to interpret the semiclassical results, i.e. how to interpolate them to finite values of spins characterising proper quantum string states.

3 Energies of quantum strings from semiclassical expansion

The standard semiclassical expansion was extensively applied to the study of energies of strings in $AdS_5 \times S^5$having large quantum numbers and thus dual to "long" SYM operators with large canonical dimensions (see, e.g., [21] for reviews). It was suggested in [11] that despite being formally valid for "large" strings with large energies and spins this expansion may be still useful also for extracting information about "small" or "slow" strings, assuming that the resulting expressions for the energies admit analytic continuation to the region of small quantum numbers such as spins. In the cases we discuss below this assumption appears to be justified, i.e. it is consistent with other sources of information about the structure of the spectrum of quantum strings in $AdS_5 \times S^5$.

Consider a classical string solution with energy E and spin J. The standard semiclassical approximation is based on expanding E in large $\sqrt{\lambda}$ with $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ kept fixed,

$$E = E\Big(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}\Big) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + \ldots \tag{3.1}$$

In the "short" (or "slow") string limit when $\mathcal{J} \ll 1$ one finds (cf. (1.6))

$$\mathcal{E}_k = \sqrt{\mathcal{J}}\ (a_{0k} + a_{1k}\mathcal{J} + a_{2k}\mathcal{J}^2 + \ldots) + \mathcal{E}_k^{(\text{nan})}\,, \tag{3.2}$$

$$\mathcal{E}_k^{(\text{nan})} = c_{0k} + c_{1k}\mathcal{J} + \ldots\ . \tag{3.3}$$

The "analytic" terms [11] written explicitly in (3.2) are the only ones present in the classical string energy and the ones that should naively appear from quantum corrections if one assumes analyticity of the string partition function in $(\text{mass})^2$ parameters of string fluctuations (this follows from the fact that $(\text{mass})^2 \sim \mathcal{J} + \mathcal{O}(\mathcal{J}^2)$ and that to obtain $\mathcal{E}_n$ from the 2d effective action one is to divide it by $\kappa \sim \sqrt{\mathcal{J}}$, see below). The "non-analytic" terms in $E_k^{(\text{nan})}$ originate from quantum "infrared" effects in the small-spin limit.

Formally, this expansion is valid for large $\sqrt{\lambda}$ and fixed $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$, i.e. $J \sim \sqrt{\lambda} \gg 1$. However, if we knew all the terms in it to arbitrary order k we could re-express $\mathcal{J}$ in terms of $J = \sqrt{\lambda}\mathcal{J}$, fix J to certain *finite* value and then re-expand E in large $\sqrt{\lambda}$ for fixed J. This is what one would need to do in order to compare with gauge-theory results for short operators in the strong coupling expansion.

Rewriting the above expansion (3.1) in terms of J we get

$$E = \sqrt{\sqrt{\lambda}J}\Big[a_{00} + \frac{a_{10}J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20}J^2 + a_{11}J + a_{02}}{(\sqrt{\lambda})^2} + \ldots\Big] + E^{(\text{nan})}\,, \tag{3.4}$$

$$E^{(\text{nan})} = c_{01} + \frac{c_{11}J + c_{02}}{\sqrt{\lambda}} + \ldots\,, \tag{3.5}$$

where a_{mk}, c_{mk} are coefficients of the k-loop string sigma model corrections. If we now set J to some finite value then in order to know, e.g., the coefficient of the $\frac{1}{(\sqrt{\lambda})^k}$ term in the square bracket in (3.4) we would need to know only a *finite number* of coefficients of up to k-loop term in the semiclassical expansion (3.2).[16]

For example, the knowledge of the 1-loop coefficient a_{01} together with the classical string energy coefficient a_{10} is sufficient to fix the $\frac{1}{\sqrt{\lambda}}$ term in the bracket in (3.4). To fix the $\frac{1}{(\sqrt{\lambda})^2}$ term, in addition to the classical and the 1-loop corrections one would need to know also the 2-loop coefficient a_{02}, etc. The same applies to the "non-analytic" part $E^{(\mathrm{nan})}$.

Fixing a specific value of J corresponding to some particular quantum string state we then end up with the strong-coupling expansion of the energy (or dimension of the corresponding "short" operator) already quoted in equations (1.6), (1.7), (1.8). This is also the same structure of the strong-coupling expansion of E as predicted by the consideration of the marginality condition of the corresponding vertex operators, see (2.1), (2.7), (2.25) (in the notation of (1.5) $a_{00}\sqrt{J} \to 2\sqrt{n-1}$, $c_{01} = b_0$, etc.).

In interpolating semiclassical expressions to finite values of spins we will need to take into account that, since we started in the region where $J \gg 1$, we should ensure that the resulting expression for the energy has the right flat-space limit as appropriate for a quantum string state with finite J; that may require to do a formal shift J by a finite amount like $J \to J - 2$.

Below we shall consider several explicit examples of expansions (3.4) for simple string solutions that can be interpolated to quantum string states that carry the same quantum numbers as some of the bosonic members of the Konishi multiplet from table 1. We will include the classical and the 1-loop string corrections and verify that, as expected, the coefficient b_1 in (1.7), (2.1) is universal, while $b_0 = c_{01}$ may change by integer shifts within the multiplet.

3.1 Small circular spinning string with $J_1 = J_2$ in S^5

We shall start with one of the simplest non-trivial string solutions in $AdS_5 \times S^5$— a rigid circular string rotating with two equal spins on an (arbitrary-size) 3-sphere inside S^5. This is one of the two $J_1 = J_2 = J$ solutions found in [7] — the one which is stable and has $J < \frac{1}{2}\sqrt{\lambda}$. The other (more well-known) one has $J \geq \frac{1}{2}\sqrt{\lambda}$ and describes a string rotating on a "big" (unit radius) S^3 of S^5 and is unstable against small perturbations.

The first (or "small-string") solution has classical energy being of the same form as in flat space, $E_0 = \sqrt{4\sqrt{\lambda}\, J}$. The second ("large-string") solution has larger energy $E_0 = \sqrt{(2J)^2 + \lambda}$ for all J apart from the "critical point" $J = \frac{1}{2}\sqrt{\lambda}$ where the two solutions coincide. While the second string is never small (it has radius of S^5) and admits a "fast-string" expansion $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1$, the first one may have an arbitrarily small radius and spin and thus has a "small-string" limit $\mathcal{J} \ll 1$ when it probes the near-flat region of S^5.

In fact, the "small-string" solution is a direct embedding into $AdS_5 \times S^5$of the following flat-space $R_t \times R^4$ solution describing a rigid circular string rotating in two orthogonal

[16]Let us stress that this is a remarkable feature of the "short string" expansion, as compared to the "long" or "fast" ($\mathcal{J} \gg 1$) string expansion considered in [7]: there the energy expressed in terms of J contained the tension $\sqrt{\lambda}$ in positive powers so to get a strong coupling expansion of the energy at fixed J one would need to resum the whole semiclassical series, i.e. that would require one to know the infinite number of semiclassical coefficients.

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planes of R^4,[17]

$$t = \kappa\tau\,, \qquad \mathrm{x}_x \equiv x_1 + ix_2 = \ a\ e^{i(\tau+\sigma)}\,, \qquad \mathrm{x}_y \equiv x_3 + ix_4 = \ a\ e^{i(\tau-\sigma)}\,, \tag{3.6}$$

$$E_{\rm flat} = \frac{\kappa}{\alpha'} = \sqrt{\tfrac{4}{\alpha'}J}\,, \qquad\qquad J_1 = J_2 = J = \frac{a^2}{\alpha'}\ . \tag{3.7}$$

Identifying the oscillator modes that are excited on this solution one may associate it with the quantum string state which is created by the following vertex operator (dots stand for the fermionic terms generally present in the superstring case)

$$e^{-iEt}\left[(\partial\mathrm{x}_x\bar\partial\mathrm{x}_x)^{\frac{J_1}{2}}(\partial\mathrm{x}_y\bar\partial\mathrm{x}_y)^{\frac{J_2}{2}} + ...\right], \qquad \alpha' E^2 = 2(J_1 + J_2 - 2)\ . \tag{3.8}$$

In the $J_1 = J_2$ case the quantum-state analog of the classical expression for the energy in (3.7) is thus found by a shift $J \to J - 1$

$$E_{\rm flat} = \sqrt{\tfrac{4}{\alpha'}(J-1)}\ . \tag{3.9}$$

Then $J_1 = J_2 = 2$ case corresponds to a state on the first massive string level $n = 2$.

Below we will be interested also in similar semiclassical string states in $AdS_5\times S^5$which in the small-string limit approach the above flat-space solution (3.7). This will allow us to relate semiclassical results to several members of the Konishi multiplet should be dual to string states at the first excited string level in the near-flat expansion of the $AdS_5 \times S^5$superstring [2, 11, 13].

There are three obvious choices for how one may embed the solution (3.6) into $AdS_5 \times S^5$:

(i) the two 2-planes may belong to S^5 leading to the $J_1 = J_2$ "small-string" solution;

(ii) the two 2-planes may belong to AdS_5 leading to a $S_1 = S_2$ "small-string" solution;

(iii) one of the 2-planes may belong to AdS_5 and the other to S^5, leading to an $S = J$ "small-string" solution.

We will discuss these three cases in turn in this and the following two subsections. Interpolated to finite values of the spins $J = 2,\ S = 2$ the corresponding string states will represent different members of the Konishi multiplet and this will allow us to verify the universality of the strong-coupling expansion of the 4-d anomalous dimension of the dual gauge theory operators.

The direct counterpart of (3.6) in $R_t \times S^5$ is described by [7][18]

$$t = \kappa\tau\ , \quad X_1 + iX_2 = a\ e^{i(\tau+\sigma)}, \quad X_3 + iX_4 = a\ e^{i(\tau-\sigma)}, \quad X_5 + iX_6 = \sqrt{1-a^2},$$
$$\mathcal{J}_1 = \mathcal{J}_2 = a^2 = \frac{\kappa^2}{4} = \mathcal{J} = \frac{J}{\sqrt{\lambda}}\,, \qquad E_0 = \sqrt{\lambda}\,\mathcal{E}_0 = \sqrt{\lambda}\,\kappa = \sqrt{4\sqrt{\lambda}J}\ . \tag{3.10}$$

[17] Here $\sigma \in [0, 2\pi)$. We shall always choose the "winding" numbers to be 1.

[18] Here X_k are the embedding coordinates of S^5, $X_1^2 + ... + X_6^2 = 1$ (i.e. we use X_k instead of n_k in (2.12)). For comparison, the "large-string" branch of the $J_1 = J_2$ solution [7] is described by $X_1 + iX_2 = \frac{1}{\sqrt2}e^{i(w\tau+\sigma)}$, $X_3 + iX_4 = \frac{1}{\sqrt2}e^{i(w\tau-\sigma)}$, $X_5 + iX_6 = 0$, where $w = 2\mathcal{J} = \sqrt{\kappa^2-1}$ is arbitrary. Notice that here we use different notation for S^5 embedding coordinates X_k as compared to n_k in (2.12).

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Remarkably, the *exact* expression for the classical energy has the same "Regge" form as in flat space (3.7) with $\frac{1}{\alpha'} \to \sqrt{\lambda}$ (we set the radius of S^5 to be 1).

The quadratic fluctuations of the $AdS_5 \times S^5$string action near this homogeneous solution were discussed in [7, 24]. Here we use the corresponding fluctuation frequencies to compute the 1-loop correction to the classical energy in (3.10). In addition to 2 massless "longitudinal" bosonic modes one finds 4 massive fluctuations in AdS_5 directions with

$$\omega_n^2 = n^2 + 4\mathcal{J} \,, \tag{3.11}$$

and 2 massless and 2 massive fluctuations in S^5, with the latter having

$$\omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}) \pm 2\sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}^2} \,. \tag{3.12}$$

The 4+4 fermionic modes have the fluctuation frequencies

$$\tilde{\omega}_{n\pm}^2 = n^2 + 1 + \mathcal{J} \pm \sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}} \,. \tag{3.13}$$

The 1-loop correction to the string energy is given by $E_1 = \frac{1}{\kappa} E_{2\mathrm{d}}$, where $E_{2\mathrm{d}}$ is determined by the logarithm of the 1-loop partition function, $E_{2\mathrm{d}} = -\frac{1}{\mathcal{T}} \ln Z_1, \ \ \mathcal{T} \to \infty$. Thus

$$E_1 = \frac{1}{\kappa} E_{2\mathrm{d}} = \frac{1}{2\sqrt{\mathcal{J}}} E_{2\mathrm{d}} \,, \qquad E_{2\mathrm{d}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \Omega_n = \frac{1}{2}\Omega_0 + \Omega_1 + \Omega_2 + \sum_{n=3}^{\infty} \Omega_n \,, \tag{3.14}$$

$$\Omega_n \equiv 4\omega_n + 2n + \omega_{n+} + \omega_{n-} - 4(\tilde{\omega}_{n+} + \tilde{\omega}_{n-}) \,. \tag{3.15}$$

Expanding in small $\mathcal{J}$ we find (we isolate $\Omega_0, \Omega_1, \Omega_2$ since the expansion of generic Ω_n is singular for $n = 0, \pm 1, \pm 2$)

$$\Omega_0 = -4 + 8\sqrt{\mathcal{J}} - 2\mathcal{J} - \mathcal{J}^2 + \dots \,, \qquad \Omega_1 = 2 - 4\sqrt{\mathcal{J}} + 5\mathcal{J} - \frac{437}{48}\mathcal{J}^2 + \dots \,, \tag{3.16}$$

$$\Omega_2 = -\frac{5}{3}\mathcal{J} + \frac{44}{27}\mathcal{J}^2 + \dots \,, \qquad \frac{1}{2}\Omega_0 + \Omega_1 + \Omega_2 = \frac{7}{3}\mathcal{J} - \frac{3445}{432}\mathcal{J}^2 + \dots \,, \tag{3.17}$$

$$\sum_{n=3}^{\infty} \Omega_n = q_1 \mathcal{J} + q_2 \mathcal{J}^2 + \dots \,, \tag{3.18}$$

$$q_1 = -\sum_{n=3}^{\infty} \frac{4}{n(n^2-1)} = -\frac{1}{3} \,, \qquad q_2 = \sum_{n=3}^{\infty} \frac{-28 + 87n^2 - 79n^4 + 8n^6}{n^3(n^2-4)(n^2-1)^3} = \frac{3121}{432} - 6\zeta(3).$$

Here, as expected, $(E_{2\mathrm{d}})_{\mathcal{J}\to 0} \to 0$ since the solution shrinks to a point in the $\mathcal{J} \to 0$ limit. Note also that the $\sqrt{\mathcal{J}}$ contributions coming from Ω_0 and Ω_1 cancel against each other, implying the absence of the constant shift c_{01} (cf. (3.4), (3.5)) in the corresponding expression for E_1. Also, the sum of Ω_n does not contain $\mathcal{J}^{3/2}$ term so there is also no "non-analytic" $\frac{c_{11}J}{\sqrt{\lambda}}$ term in E_1 (cf. (3.3), (3.5)).

Explicitly, we find (cf. (3.4))

$$\mathcal{E}_1 = \sqrt{\mathcal{J}} + a_{11}\mathcal{J}^{3/2} + \mathcal{O}(\mathcal{J}^{5/2}) \,, \qquad a_{11} = -\frac{3}{8} - 3\zeta(3) \,, \tag{3.19}$$

$$E = E_0 + E_1 = \sqrt{4\sqrt{\lambda}J}\left[1 + \frac{1}{2\sqrt{\lambda}} + \frac{a_{11}J}{2\lambda} + \mathcal{O}\Big(\frac{J^2}{\lambda^{3/2}}\Big)\right] \,, \qquad E_1^{(\mathrm{nan})} = 0 \,. \tag{3.20}$$

This result is formally valid in the limit when $\sqrt{\lambda}$ is first taken to be large for fixed $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ and *then* $\mathcal{J}$ is taken to be small so that $J \ll \sqrt{\lambda}$. However, as discussed above, we may formally try to interpolate it to finite values of J. In that case the $\frac{J}{\lambda}$ term in (3.20) is of the same order as a 2-loop correction which we will not compute and so we should ignore it here.

The same applies to the non-analytic term: there might in principle be 2-loop correction producing c_{02} term in (3.5) which is of the same order as the (absent) 1-loop $\frac{c_{11}J}{\sqrt{\lambda}}$ term; we find it very unlikely that $c_{02} \neq 0$. Thus we conjecture that b_2 in (1.8) should be zero.

Comparing (3.20) with the flat-space energy of the quantum string state (3.8) corresponding to the classical solution (3.6), (3.7), i.e. with (3.9), we conclude that in order to interpret (3.20) as a quantum string energy we should shift J as in (3.9), i.e. $J \to J-1$. Then

$$E = 2\sqrt{\sqrt{\lambda}(J-1)}\Big[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{J}{\lambda}\Big)\Big] . \tag{3.21}$$

Setting now $J = 2$ we end up with

$$E = 2\sqrt[4]{\lambda}\Big[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{1}{\lambda}\Big)\Big] . \tag{3.22}$$

The reason for this choice of $J = J_1 = J_2 = 2$ is that such a state belongs to the first excited string level and the corresponding representation (1.1) $(E, 0, 0; 2, 2, 0)$ or in Dynkin label notation (1.2) $[2, 0, 2]_{(0,0)}$ is present in the table 1 of supersymmetry descendants of the singlet Konishi operator $\mathrm{Tr}(\bar{\Phi}_k \Phi_k)$. Indeed, there is one of such states at each of the levels $\ell = 4$ (with $\Delta_0 = 2 + \frac{1}{2}\ell = 4$), $\ell = 8$ ($\Delta_0 = 6$) and $\ell = 12$ ($\Delta_0 = 8$), i.e.

$$[2, 0, 2]_{(0,0)} \ : \qquad \Delta_0 = 4 \ (1) \ ; \qquad \Delta_0 = 6 \ (1) \ ; \qquad \Delta_0 = 8 \ (1) \ . \tag{3.23}$$

The $\Delta_0 = 4$ Konishi state is represented by the operator $\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ from the su(2) sector of the SYM theory.

According to (3.22), the universal coefficient b_1 in (2.7) should then be equal to 1. It is not clear a priori which of the three states in (3.23) should be described by the above semiclassical $J_1 = J_2$ string; the corresponding dimensions are expected to be different only by the constant Δ_0 term in (2.7). Since the above circular solution appears to have lowest energy for given spins we shall conjecture that it represents the lowest-dimension state with $\Delta_0 = 4$. In this case the value of $b_0 = 0$ in (3.22) (cf. (1.8), (2.1)) translates into

$$\mathrm{b}_0 = -4 \, , \tag{3.24}$$

as already quoted in (1.9). Further evidence for these values of b_1 and b_0 will be provided below.

3.2 Small circular spinning string with $S_1 = S_2$ in AdS^5

As another closely related example let us now consider the counterpart of the flat-space solution (3.6) when the circular spinning string rotates solely in AdS_5 [7, 23]. In terms of the AdS_5 embedding coordinates Y_a we get (in the conformal gauge)[19]

$$Y_0 + iY_5 = \sqrt{1 + 2r^2}\ e^{i\kappa t} \, , \quad Y_1 + iY_2 = r\ e^{i(w\tau + \sigma)} \, , \quad Y_3 + iY_4 = r\ e^{i(w\tau - \sigma)} \ . \tag{3.25}$$

[19] Here $Y_0^2 + Y_5^2 - Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = 1$; again, we use different notation for the embedding coordinates than in (2.12): Y_a instead of N_a.

Here $r = \sinh\rho_0 = \frac{1}{4}\kappa^2$, $w^2 = \kappa^2 + 1$ and the energy and the spins are given by

$$E_0 = \sqrt{\lambda}\,\mathcal{E}_0, \quad S_1 = S_2 = S = \sqrt{\lambda}\,\mathcal{S}, \quad \mathcal{S} = \frac{1}{4}\kappa^2\sqrt{\kappa^2+1}, \quad \mathcal{E}_0 = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{\kappa^2+1}}\ . \tag{3.26}$$

This solution again admits a "small-string" limit ($\mathcal{S} \to 0$)[20] in which it represents a small circular string rotating in two orthogonal planes around its c.o.m. in the central near-flat region of AdS_5. Its flat-space limit is thus again given by (3.6).

In the $\mathcal{S} = \frac{S}{\sqrt{\lambda}} \ll 1$ expansion

$$\kappa = 2\sqrt{\mathcal{S}} - 2\mathcal{S}^{3/2} + 9\mathcal{S}^{5/2} + \ldots\ , \tag{3.27}$$

and expressed in terms of $S = \sqrt{\lambda}\mathcal{S}$ the classical energy becomes [7] (cf. (3.2), (3.4))

$$E_0 = 2\sqrt{\sqrt{\lambda}S}\left[1 + \frac{S}{\sqrt{\lambda}} - \frac{3S^2}{2\lambda} + \mathcal{O}\Big(\frac{S^3}{\lambda^{3/2}}\Big)\right]. \tag{3.28}$$

Here in contrast to the $J_1 = J_2$ solution (3.10) the classical energy contains non-trivial "curvature" corrections which modify the leading-order flat-space "Regge" behavior.

The 1-loop correction to the energy of this solution was computed in [22]. Expanding the fluctuation frequencies in small $\mathcal{S}$ it is straightforward to find the corresponding analogs of (3.19), (3.20). In addition to 5+2 massless modes (2 of which are canceled by the conformal-gauge ghosts) there are 3 non-trivial massive AdS_5 fluctuation modes with the characteristic frequencies $\omega_n^{(i)}$ ($i = 1,2,3$) given by the solutions of the cubic equation [22]

$$\omega_n^6 + c_1\omega_n^4 + c_2\omega_n^2 + c_3 = 0\,, \qquad c_1 = -8 - 10\kappa^2 - 3n^2\,, \tag{3.29}$$

$$c_2 = 16 + 40\kappa^2 + 24\kappa^4 + 8\kappa^2 n^2 + 3n^4\,, \qquad c_3 = -n^2(n^2-4)(n^2-4-2\kappa^2)\ . \tag{3.30}$$

The 4+4 fermionic frequencies are [22]

$$\tilde{\omega}_{n\pm}^2 = n^2 + 1 + \frac{5}{4}\kappa^2 \pm \sqrt{4n^2 + \kappa^2 + 3n^2\kappa^2 + \kappa^4}\ . \tag{3.31}$$

Then the analog of (3.14) is

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Omega_n\,, \qquad \Omega_n = 5n + \omega_n^{(1)} + \omega_n^{(2)} + \omega_n^{(3)} - 4(\tilde{\omega}_{n+} + \tilde{\omega}_{n-})\ . \tag{3.32}$$

The $\mathcal{S} \to 0$ expansion gives (cf. (3.17), (3.18))[21]

$$\frac{1}{2}\Omega_0 + \Omega_1 + \Omega_2 = -4\sqrt{\mathcal{S}} - \frac{7}{3}\mathcal{S} + 4\mathcal{S}^{3/2} + \ldots\ , \tag{3.33}$$

$$\sum_{n=3}^{\infty}\Omega_n = \sum_{n=3}^{\infty}\frac{4}{n(n^2-1)}\,\mathcal{S} + \mathcal{O}(\mathcal{S}^2) = \frac{1}{3}\mathcal{S} + \mathcal{O}(\mathcal{S}^2)\ . \tag{3.34}$$

Again, $E_{2\mathrm{d}} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\Omega_n$ vanishes in the $\mathcal{S} \to 0$ limit when the string shrinks to a point. However, in contrast to the case of $J_1 = J_2$ solution here $E_{2\mathrm{d}}$ approaches zero as a

[20] This solution is stable for $\mathcal{S} \leq 1.17$ [7, 22].

[21] The $\mathcal{S} \to 0$ expansion of $\sum_{n=3}^{\infty}\Omega_n$ contains only integer powers of $\mathcal{S}$.

square root of spin instead of linear function of spin, i.e. naively there is a "non-analytic" contribution coming from $\mathcal{S}^{1/2}$ (and $\mathcal{S}^{3/2}$) term in (3.33). Dividing by κ in (3.32) and using (3.27) appears to lead to[22]

$$E_1(?) = -2 - \sqrt{\mathcal{S}} + \mathcal{O}(\mathcal{S}) \ . \tag{3.35}$$

However, a more careful analysis described in appendix implies that this -2 constant shift is an artifact of the procedure of representing the 1-loop correction as a sum of characteristic frequencies and expanding each frequency in small $\mathcal{S}$ separately. Computing the 1-loop correction to 2-d energy as a combination of logarithms of determinants of the quadratic fluctuation operators and then expanding the result in small $\mathcal{S}$ leads actually to the vanishing result for the coefficient of the leading non-analytic term $\sqrt{\mathcal{S}}$ in E_{2d}. Then instead of (3.35) one finds

$$E_1 = -\sqrt{\mathcal{S}} + \mathcal{O}(\mathcal{S}) \, , \tag{3.36}$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda} S}\left[1 + \frac{S - \frac{1}{2}}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{S^2}{\lambda}\Big)\right] + \mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big) \ . \tag{3.37}$$

Notice that the leading 1-loop term in the $S_1 = S_2$ case (3.36) differs from the leading 1-loop term in the $J_1 = J_2$ (3.19) only by a sign and $\mathcal{J} \to \mathcal{S}$. One may try to attribute this sign difference to the difference in the sign of the curvature of S^5 and of AdS_5.

As in the case of the small $J_1 = J_2$ string, the flat-space counterpart of this solution (3.6) corresponds to the quantum string state associated to (3.8) with $J_i \to S_i$ and $S_1 = S_2$. Then S in (3.37) should be redefined $S \to S-1$ to match the flat-space limit (3.9) (cf. (3.38))

$$E = 2\sqrt{\sqrt{\lambda}(S-1)}\left[1 + \frac{(S-1) - \frac{1}{2}}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{S^2}{\lambda}\Big)\right] + \mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big) \ . \tag{3.38}$$

This suggests that in the case of $S = 2$, i.e. the corresponding string state belonging to the first excited level, the strong-coupling expansion of its energy should thus be

$$E = 2\sqrt[4]{\lambda}\left[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{1}{\lambda}\Big)\right] + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big) \ . \tag{3.39}$$

Here the subleading $\mathcal{O}(\frac{1}{\lambda})$ term in the bracket and the last "non-analytic" term $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$ term are sensitive to the 2-loop string corrections and thus beyond our reach. Remarkably, the two leading strong-coupling terms in (3.39) are exactly the same as in (3.22) found above for the $J_1 = J_2 = 2$ string state.

This is perfectly consistent with the expectation that the $S_1 = S_2 = 2$ state or $(E, 2, 2; 0, 0, 0)$ should also belong to the Konishi multiplet and thus should have the same anomalous dimension as the state $(E, 0, 0; 2, 2, 0)$ represented by the $J = 2$ limit of the $J_1 = J_2$ solution. Indeed, in the Dynkin-label notation (1.2) this state corresponds to $[0, 0, 0]_{(2,0)}$ and there are two of such states in the Konishi multiplet table 1 (cf. (3.23))

$$[0, 0, 0]_{(2,0)} \ : \qquad \Delta_0 = 4 \ (1) \ ; \qquad \Delta_0 = 8 \ (1) \ . \tag{3.40}$$

[22]This expression was independently found by A. Tirziu.

The corresponding gauge theory operator with $\Delta_0 = 4$ is $\mathrm{Tr}([D_{1+i2}, D_{3+i4}])^2$ or $\mathrm{Tr}(F_{1+i2,3+i4})^2$.[23]

It is natural to assume again that the $S_1 = S_2 = 2$ string state correspond to the Konishi multiplet member with $\Delta_0 = 4$. Then the resulting values of b_0 and b_1 as predicted by (3.39) are the same as in (1.9), (3.24).[24]

3.3 Small circular spinning string with $S = J$ in $AdS_5 \times S^5$

Another embedding of the 2-spin flat-space solution (3.6) into $AdS_5 \times S^5$is found by considering one spinning plane being in AdS_5 and another — in S^5. The well-known rigid circular (S, J) solution of this type [23, 25] where the string in S^5 is wrapped on a big circle, does not, however, admit a "small-string" limit in which the classical energy takes the flat-space Regge form (3.7). However, it is easy to construct its close relative that does have the required limit.

To achieve this one is to put the circular string on a 2-sphere of an arbitrary radius inside S^5. In terms of the AdS_5 and S^5 embedding coordinates we then get (cf. (3.10), (3.25))

$$Y_0 + iY_5 = \sqrt{1 + r^2}\ e^{i\kappa t}\,, \qquad Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)}\,, \qquad w^2 = \kappa^2 + 1\,, \tag{3.41}$$

$$X_1 + iX_2 = a\ e^{i(\tau-\sigma)}\,, \qquad X_3 + iX_4 = \sqrt{1 - a^2}\ . \tag{3.42}$$

Here $r = \sinh\rho_0$ and $a = \sin\gamma_0$ determine the size of the string in AdS_5 and S^5 respectively. The conformal gauge conditions imply

$$(1 + r^2)\kappa^2 = r^2(w^2 + 1) + 2a^2\,, \qquad r^2 w = a^2\ . \tag{3.43}$$

Thus for this solution one has $\mathcal{S} = r^2 w = \mathcal{J} = a^2 \le 1$, i.e. $S = J \le \sqrt{\lambda}$. Also, $\mathcal{E}_0 = (1+r^2)\kappa = \kappa + \frac{\mathcal{S}\kappa}{\sqrt{\kappa^2+1}}$, where κ satisfies the equation $\kappa^2 = \frac{2\mathcal{S}}{\sqrt{\kappa^2+1}} + 2\mathcal{S}$ which is readily solved.

Explicitly, we find (cf. (3.27), (3.28))

$$\kappa = \sqrt{\sqrt{\frac{1}{4} + 2\mathcal{S}} - \frac{1}{2} + 2\mathcal{S}} = 2\sqrt{\mathcal{S}} - \mathcal{S}^{3/2} + \frac{15}{4}\mathcal{S}^{5/2} + \ldots\,, \tag{3.44}$$

$$\mathcal{E}_0 = \sqrt{\sqrt{\frac{1}{4} + 2\mathcal{S}} - \frac{1}{2} + 2\mathcal{S}}\ \left(1 + \frac{\mathcal{S}}{\sqrt{\sqrt{\frac{1}{4} + 2\mathcal{S}} + \frac{1}{2} + 2\mathcal{S}}}\right) = 2\sqrt{\mathcal{S}} + \mathcal{S}^{3/2} + \ldots \tag{3.45}$$

$$E_0 = \sqrt{\lambda}\mathcal{E}_0 = 2\sqrt{\sqrt{\lambda}S}\ \left[1 + \frac{S}{2\sqrt{\lambda}} - \frac{5S^2}{8\lambda} + \mathcal{O}\Big(\frac{S^3}{\lambda^{3/2}}\Big)\right]\ . \tag{3.46}$$

In the small-size or $\mathcal{S} = \mathcal{J} \to 0$ limit (when $w \to 1,\ r \to a \to 0$) this solution reduces to the flat-space one (3.6) with the energy taking the form (3.7).

---

[23] It belongs to a family of field-strength operators [26] conjectured in [22] to be related to $S_1 = S_2$ semiclassical strings.

[24] As for the value of b_2, as already mentioned it receives contribution both from the 1-loop $\frac{c_{11}S}{\sqrt{\lambda}}$ term and 2-loop term $\frac{c_{02}}{\sqrt{\lambda}}$ (cf. (3.5)), and their sum may vanish due to underlying supersymmetry of the theory, as suggested by the remarks we made in the context of the vertex operator approach in section 2.

At the $\mathcal{S} = \mathcal{J} = 1$ point (where $a = 1,\ \kappa = \sqrt{3},\ w = 2,\ r = \sqrt{2}$) this "small-string" $S = J$ solution coincides with the "large-string" $S = J$ solution discussed in [23, 25].[25]

To compute the 1-loop correction to the energy of this solution it turns out to be more efficient to use the path integral approach in which (see appendix)

$$E_1 = \frac{1}{2\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=-\infty}^{\infty}\ln\frac{P_B(\omega,n,\mathcal{S})}{P_F(\omega,n,\mathcal{S})}\ . \tag{3.47}$$

Here P_B and P_F are, respectively, the bosonic and fermionic characteristic polynomials, i.e. the equations $P_B(\omega,n,\mathcal{S}) = 0$ and $P_F(\omega,n,\mathcal{S}) = 0$ determine the characteristic frequencies. $P_B(\omega,n,\mathcal{S})$ is found to be

$$\begin{aligned} P_B =\ &(\omega-n)^5(\omega+n)^6\Big[(\omega-n)^2-4(1-\mathcal{S})\Big]\Big[\omega^2-n^2+\frac{1}{2}(1-4\mathcal{S}-\sqrt{1+8\mathcal{S}})\Big]^2\\ &\times\Big[(\omega-n)[(\omega+n)^2-4]+(3-8\mathcal{S}-3\sqrt{1+8\mathcal{S}})\omega-(1-\sqrt{1+8\mathcal{S}})n\Big]\ . \end{aligned} \tag{3.48}$$

The fermionic characteristic polynomial is more complicated and we will give only the first few terms in its expansion in $\mathcal{S}$:

$$\begin{aligned} P_F =\ &[\omega^2-(n+1)^2]^3\ [\omega^2-(n-1)^2]^3\\ &\times\Big[[\omega^2-(n+1)^2][(\omega^2-(n-1)^2]+(1-3\omega^2+4\omega n+n^2)\mathcal{S}\Big]+\mathcal{O}(\mathcal{S}^2)\ . \end{aligned} \tag{3.49}$$

At the next order in the small $\mathcal{S}$ expansion the three-fold degeneracy is lifted to a two-fold one. We have checked that at the junction point $\mathcal{S} = \mathcal{J} = 1$ the characteristic frequencies following from the equations $P_B = 0$, $P_F = 0$ reproduce the ones of the "large-string" (S, J) solution found in [25].

While the characteristic polynomials are naturally functions of $\mathcal{S}$, their roots, for low mode numbers ($n = -1, 0, 1$), turn out to depend on $\sqrt{\mathcal{S}}$ for small $\mathcal{S}$. It is therefore important to analyze these modes separately. A short calculation shows that the $\mathcal{S}$ dependence

[25] For completeness, let us recall the form of the "large-string" solution of [23] (as above, we assume that the two possible winding numbers are equal to 1): $Y_0 + iY_5 = \sqrt{1+r^2}\, e^{i\kappa t}$, $Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)}$, $X_1 + iX_2 = e^{i(\omega\tau-\sigma)}$, where $w^2 = \kappa^2 + 1 \geq 1$, $\mathcal{S} = r^2 w = \omega = \mathcal{J}$. Then $\mathcal{E}_0 = \kappa + \frac{\mathcal{S}\kappa}{\sqrt{\kappa^2+1}}$, where $\kappa(\mathcal{S})$ satisfies $\kappa^2 = \frac{2\mathcal{S}}{\sqrt{\kappa^2+1}} + \mathcal{S}^2 + 1$. This cubic equation for κ^2 admits two real solutions for κ (third one is unphysical): $\kappa^{(1,2)} = \sqrt{1 + \frac{1}{2}\mathcal{S}^2 \pm \frac{1}{2}\mathcal{S}\sqrt{8+\mathcal{S}^2}}$. The first solution is defined for any $\mathcal{S} \geq -1$, and the second — for any $\mathcal{S} \leq 1$. The corresponding energies are

$$\mathcal{E}_0^{(1,2)} = \sqrt{1+\frac{1}{2}\mathcal{S}^2 \pm \frac{1}{2}\sqrt{8+\mathcal{S}^2}}\ \left[1 + \frac{\mathcal{S}}{\sqrt{2+\frac{1}{2}\mathcal{S}^2 \pm \frac{1}{2}\sqrt{8+\mathcal{S}^2}}}\right] .$$

Only the first branch which admit the large $\mathcal{S}$ expansion, $\mathcal{E}_0^{(1)} = 2\mathcal{S} + \frac{1}{\mathcal{S}} - \frac{5}{4\mathcal{S}^3} + \ldots$, was considered in [23, 25] (where the existence of this simple analytic expressions for the energy was not noticed). In the small $\mathcal{S}$ expansion we get $\mathcal{E}_0^{(1)} = 1 + \sqrt{2}\ \mathcal{S} + \frac{\mathcal{S}^2}{4} - \frac{\mathcal{S}^3}{8\sqrt{2}} + \ldots$ and $\mathcal{E}_0^{(2)} = 1 - \frac{\mathcal{S}^2}{4} - \frac{\mathcal{S}^3}{4\sqrt{2}} + \ldots$. This solution thus does not have the flat-space Regge asymptotics; this is not surprising since here the string is wrapped on a big circle of S^5 and its tension gives large contribution to the energy even for small spin. At the limiting point $\mathcal{S} = \mathcal{J} = 1$ the above "small-string" solution (3.41) goes over to the first branch of the "large-string" solution; in particular, both energies become equal $\mathcal{E}_0 = \mathcal{E}_0^{(1)} = \frac{3\sqrt{3}}{2}$ (while $\mathcal{E}_0^{(2)} = 0$ at $\mathcal{S} = 1$). For $0 < \mathcal{S} < 1$ the energy of the "small-string" solution is always smaller than that of the "large-string" one.

of the contribution of the low-lying modes is

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \frac{P_B(\omega,-1,\mathcal{S})}{P_F(\omega,-1,\mathcal{S})} \frac{P_B(\omega,0,\mathcal{S})}{P_F(\omega,0,\mathcal{S})} \frac{P_B(\omega,+1,\mathcal{S})}{P_F(\omega,+1,\mathcal{S})} = \mathcal{O}(\mathcal{S}) \,, \tag{3.50}$$

i.e. it does not yield an $\sqrt{\mathcal{S}}$-dependent leading term.

Explicitly, the leading $\mathcal{S}$ dependence of the corresponding part of the integrand in (3.47), i.e. the integrand of (3.50), is thus found to be linear in $\mathcal{S}$

$$-\frac{72(\omega^2+1)}{(\omega^2-1)(\omega^2-4)(\omega^2-9)}\,\mathcal{S} \quad . \tag{3.51}$$

The denominators here may be associated to propagators of various modes of the world-sheet theory. This defines the correct treatment of the ω-integral around these poles to be given by the usual $i\epsilon$ prescription; equivalently, we may just "Wick-rotate" the integrand, using the fact that it decays sufficiently fast at large ω. As a result, the ω integral of (3.51) turns out to vanish identically.

The leading small $\mathcal{S}$ dependence of a generic term in the sum in (3.47) may also be extracted by expanding the integrand. For a generic term with $|n| \geq 2$ we get

$$-\frac{8\left[3\omega^6+5\omega^4(3n^2-1)-\omega^2(15n^4-76n^2+32)-(n^2-4)^2(3n^2-1)\right]}{[\omega^2-(n-2)^2][\omega^2-(n-1)^2](\omega^2-n^2)[\omega^2-(n+1)^2][\omega^2-(n+2)^2]}\,\mathcal{S} \quad . \tag{3.52}$$

The apparent small ω singularity at $n=\pm 2$ is, in fact, cured by the numerator, which is proportional to ω^2 at those points. The absence of singularities in the integration domain of ω justifies this term-by-term expansion and confirms the absence of lower-order terms which are non-analytic in $\mathcal{S}$.

Defining the integral through "Wick rotation" as discussed in the case of the $n = -1, 0, 1$ modes, implies that the integral of (3.52) also vanishes identically. This vanishing may be confirmed by the direct analysis of the sum of the characteristic frequencies for $|n| \geq 2$ (cf. (3.18), (3.34)).

All this implies that the leading term in the small $\mathcal{S}$ expansion of the integral in (3.47) is proportional to $\mathcal{S}^{3/2}$; after dividing by $\kappa = 2\sqrt{\mathcal{S}} + \ldots$ in (3.44) we conclude that here (cf. (3.19), (3.36))

$$E_1 = \mathcal{O}(\mathcal{S}) = \mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big) \,. \tag{3.53}$$

One may try to attribute the cancellation of the leading $\sim \sqrt{\mathcal{S}}$ 1-loop correction to the cancellation between the AdS_5 and S^5 contributions (recall the opposite signs of the 1-loop $\sqrt{\mathcal{J}}$ term in the $J_1 = J_2$ (3.19) and the 1-loop $\sqrt{\mathcal{S}}$ term in the $S_1 = S_2$ (3.36) cases).

We conclude that the two leading terms in the 1-loop corrected energy of the small rigid circular $S = J$ string rotating both in AdS_5 and S^5 are given simply by the classical expression (3.46). For the corresponding quantum string state in the near-flat limit we should find, as in (3.8), $\alpha' E^2 = 2(S + J - 2) = 4(S - 1)$. Shifting $S \to S - 1$ to have the correct flat-space limit we end up with (cf. (3.21), (3.38))

$$E = 2\sqrt{\sqrt{\lambda}(S-1)}\Big[1 + \frac{(S-1)}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{S^2}{\lambda}\Big)\Big] + \mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big) \,. \tag{3.54}$$

Since we are interested in a state in the Konishi multiplet that should belong to the first excited string level we should interpolate this result to $S = J = 2$. Remarkably, the leading coefficients in (3.54) for $S = 2$ are again the same as in the $J_1 = J_2 = 2$ (3.21), (3.22) and $S_1 = S_2 = 2$ (3.38), (3.39) cases discussed above:

$$E = 2\sqrt[4]{\lambda}\Big[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{1}{\lambda}\Big)\Big] + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big) \ . \tag{3.55}$$

This is again consistent with the expectation that all these states should belong to the same supermultiplet, so their energies may differ only by a constant λ-independent shifts.

The representation corresponding to the $S = J = 2$ state is $(E, 2, 0; 2, 0, 0)$ or in the Dynkin label notation $[0, 2, 0]_{(1,1)}$. There are 1+3+1 such states present in the Konishi multiplet table 1 (cf. (3.23), (3.40))

$$[0, 2, 0]_{(1,1)} \ : \qquad \Delta_0 = 4\ (1)\ ; \qquad \Delta_0 = 6\ (3)\ ; \qquad \Delta_0 = 8\ (1)\ . \tag{3.56}$$

The dual gauge theory operator at level $\Delta_0 = 4$ is the familiar one from the sl(2) sector: $\mathrm{Tr}[\Phi_1(D_{1+i2})^2\Phi_1]$.

We shall assume again that the $S = J = 2$ state is dual to the lowest-dimension $\Delta_0 = 4$ state in the Konishi multiplet. Then in addition to $b_1 = 1$ as implied by (3.55) we again find $\mathrm{b}_0 = -4$ as in the two previous cases.

3.4 Small folded spinning strings in $AdS_5 \times S^5$

One may also consider other semiclassical solutions in $AdS_5 \times S^5$that in the small spin limit reduce to flat-space solutions that may be interpreted as representing massive string states. One familiar example is the rigid folded string in AdS_5 with spin S [27, 28]. There is a similar folded string solution in S^5 with spin J [27]. One may also consider their generalization when folded string is rotating both in AdS_5 and S^5 with spins $S = J$. Interpolated to $S = 4$, $J = 4$ and $S = J = 2$ respectively these configurations should represent different states at the first excited string level and thus should be dual to different states in the Konishi multiplet.

As we shall discuss below, the 1-loop corrected energy for the corresponding $AdS_5 \times S^5$solutions when interpolated to the respective values of the spins reproduces the same expression (3.22), (3.39), (3.55) as found above in the case of the circular string examples. This provides further evidence of the consistency of the suggested picture.

3.4.1 Folded string with spin S in AdS_5

The small spin limit of the classical energy of the folded spinning string in AdS_5 has the expected behavior $E_0 = \sqrt{2\sqrt{\lambda}S} + \ldots$. The small-spin expansion of the 1-loop correction to its energy is more complicated to compute than in the homogeneous string examples considered above as here the solution involves elliptic functions. This problem was first addressed in [11] and then also discussed in an unpublished work in [29, 30].[26] The general

[26] A generalization to include dependence on the string center-of-mass momentum J in S^5 was considered in [31].

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structure of the quantum-corrected energy found in the semiclassical expansion ($\sqrt{\lambda} \gg 1$, $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$=fixed) and then expanded in $\mathcal{S} \to 0$ is the same as in (3.4) (with J replaced by S) [11, 30]

$$E = \sqrt{2\sqrt{\lambda}S}\left[1+\frac{\frac{3}{8}S+a_{01}}{\sqrt{\lambda}}+\frac{-\frac{21}{128}S^2+a_{11}S+a_{02}}{\lambda}+\mathcal{O}\Big(\frac{S^3}{(\sqrt{\lambda})^3}\Big)\right]+E^{(\text{nan})}\,, \tag{3.57}$$

$$E^{(\text{nan})} = c_{01} + \frac{c_{11}S + c_{02}}{\sqrt{\lambda}} + \ldots\,, \tag{3.58}$$

where the coefficients a_{01}, a_{11}, c_{01} are the 1-loop ones, a_{02}, c_{11} are the 2-loop one, etc. The 1-loop values found in [11] were $a_{01} = 3-4\ln 2 = 0.227$, $\ a_{11} = -\frac{1219}{576}+\frac{3}{2}\ln 2-\frac{3}{4}\zeta(3)$.[27] An alternative computation of the leading 1-loop coefficient a_{01} in [29] (based on extracting the fluctuation spectrum from the algebraic curve description [32]) led to a different numerical result $a_{01} \approx -0.25$. Due to some uncertainties in the treatment of the zero modes in the original computation in [11], here we shall assume that the result of [29] is actually the right one, and in fact, is exactly given by[28]

$$a_{01} = -\frac{1}{4}\ . \tag{3.59}$$

Also, the analysis [30] of the separate zero-mode contributions (coming from the mixed bosonic modes in AdS_5) appears to give

$$c_{01} = 2\ . \tag{3.60}$$

Assuming the validity of (3.59) and (3.60) the classical plus the 1-loop result for the energy is then found to be

$$E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}S}\left[1+\frac{\frac{3}{8}S-\frac{1}{4}}{\sqrt{\lambda}}+\mathcal{O}\Big(\frac{S^2}{\lambda}\Big)\right]+2+\mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big)\ . \tag{3.61}$$

The flat-space limit of this solution (cf. (3.6))

$$t = \kappa\tau\,, \qquad \mathrm{x}_1 \equiv x_1 + ix_2 = \ a\sin\sigma\ e^{i\tau}\,, \qquad E_{\text{flat}} = \sqrt{\frac{2}{\alpha'}S}\,, \qquad S = \frac{a^2}{2\alpha'}\,, \tag{3.62}$$

is a semiclassical counterpart of the quantum string state on the leading Regge trajectory represented by the vertex operator (cf. (3.8))[29]

$$e^{-iEt}\left[(\partial\mathrm{x}_x\bar\partial\mathrm{x}_x)^{\frac{S}{2}}+\ldots\right], \qquad \alpha' E^2 = 2(S-2)\ . \tag{3.63}$$

[27] It is interesting to note that the presence of $\zeta(3)$ in the a_{11} coefficient appears to be a universal feature — it is also present in the case of the $J_1 = J_2$ string in (3.19), (3.20). It should thus appear in the next-to-next-to leading coefficient b_3 in the strong-coupling expansion (1.5), (2.7) of the anomalous dimension of the Konishi operator.

[28] At this order of perturbation theory the reasoning based on what one should expect to find by computing the anomalous dimensions of the corresponding vertex operators suggests that this coefficient should be expressed in terms of rational numbers only.

[29] The corresponding (bosonic) Fock space state is $(a_1^\dagger \tilde a_1^\dagger)^{\frac{S}{2}}|0, E>$. The semiclassical string is represented in this Fock space as a coherent state $\exp(\sqrt{S}a_1^\dagger + \sqrt{S}\tilde a_1^\dagger)|0, E>$.

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To be able to continue (3.61) to small values of S and match the correct flat-space limit one should shift $S \to S-2$, thus getting (cf. (3.21), (3.38), (3.54))

$$E = \sqrt{2\sqrt{\lambda}(S-2)}\left[1 + \frac{\frac{3}{8}(S-2) - \frac{1}{4}}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{S^2}{\lambda}\Big)\right] + 2 + \mathcal{O}\Big(\frac{S}{\sqrt{\lambda}}\Big) \ . \tag{3.64}$$

Then for the state on the first excited string level, i.e. for $S=4$, we finish with

$$E = 2\sqrt[4]{\lambda}\left[1 + \frac{1}{2\sqrt{\lambda}} + \mathcal{O}\Big(\frac{1}{\lambda}\Big)\right] + 2 + \mathcal{O}\Big(\frac{1}{\sqrt{\lambda}}\Big) \ . \tag{3.65}$$

Remarkably, the first two leading terms here are exactly the same as in all the three of the above circular string cases, (3.22), (3.39), (3.55).

This is how it should be as the $S=4$ state should also belong to the Konishi multiplet and thus should have the same anomalous dimension. The corresponding representation is $(E, 4, 0; 0, 0, 0)$ or $[0,0,0]_{(2,2)}$ and there is indeed just one such state in the Konishi multiplet table 1 (cf. (3.23), (3.40), (3.56))[30]

$$[0,0,0]_{(2,2)} \quad : \qquad \Delta_0 = 6 \ (1) \ . \tag{3.66}$$

Since this state has $\Delta_0 = 6$, the constant shift $b_0 = 2$ in (3.65) (cf. (1.8)) is then perfectly consistent with the value of $\mathrm{b}_0 = -4$ in (1.9), (3.24).

3.4.2 Folded string with spin J in S^5

Similarly to case of the flat-space circular string (3.6) that can be embedded either in S^5 or in AdS_5 (or both) we can also embed the flat-space folded string (3.62) not in AdS_5 but in S^5. The corresponding solution [27] is the direct analog of the one in AdS_5.[31] In that case the classical energy has the following small $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ expansion

$$E_0 = \sqrt{2\sqrt{\lambda}J}\left[1 + \frac{\frac{1}{8}J}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{J^2}{\lambda}\Big)\right] \ . \tag{3.67}$$

While the 1-loop correction in this case was not computed so far, we shall conjecture that the coefficient a_{01} in the analog of (3.57) here should have the *opposite* sign compared to (3.59) since the sign of the curvature of S^5 is opposite to that of AdS_5. Indeed, as we have seen on the examples of the $J_1 = J_2$ and $S_1 = S_2$ circular string solutions, the respective 1-loop coefficients in (3.21) and (3.38) differ only by the sign. We shall thus assume that for the folded string in S^5 one should get $a_{01} = \frac{1}{4}$. We shall also assume that the constant c_{01} in the corresponding analog of the "non-analytic" part of the 1-loop energy (3.58) should be again given by 2 as in (3.60).

Taking also into account the shift $J \to J-2$ to match the required flat-space limit we can then generalize (3.67) to the following 1-loop corrected result (cf. (3.61))

$$E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}(J-2)}\ \left[1 + \frac{\frac{1}{8}(J-2) + \frac{1}{4}}{\sqrt{\lambda}} + \mathcal{O}\Big(\frac{J^2}{\lambda}\Big)\right] + 2 + \mathcal{O}\Big(\frac{J}{\lambda}\Big) \ . \tag{3.68}$$

[30] The dual SYM operator should contain terms like $\bar{\Phi}_k (D_{1+i2})^4 \Phi_k$.

[31] Explicitly, in terms of the embedding coordinates of S^2 inside of S^5 we have $X_1 + iX_2 = \sin\psi(s)\, e^{iw\tau}$, $X_3 = \cos\psi(s)$, $\psi'^2 + w^2 \sin^2\psi = \kappa^2$.

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We observe that for the state with $J = 4$ which is at the first excited level the value of (3.68) is the same as in (3.65), i.e. gives $b_1 = 1$ as in all other cases discussed above.

The state with $J = 4$ is in the representation $(E, 0, 0; 4, 0, 0)$ or $[0, 4, 0]_{(0,0)}$; there is just one such state in the Konishi table 1 (cf. (3.23), (3.66)):[32]

$$[0,4,0]_{(0,0)} \quad : \qquad \Delta_0 = 6 \ (1) \ . \tag{3.69}$$

As in the previous folded string example, the $b_0 = \Delta_0 + \mathrm{b}_0 = 2$ is then again consistent with $\mathrm{b}_0 = -4$.

3.4.3 Folded string with two spins $S = J$ in $AdS_5 \times S^5$

Finally, as in the third "mixed" embedding of the circular string in $AdS_5 \times S^5$discussed in the subsection 3.3, we may consider also another (S, J) solution given by the direct superposition of the folded strings rotating in AdS_5 and in S^5 "glued" together by the Virasoro condition. Here the leading terms in the small-spin expansion of the classical energy take the expected "direct superposition" form (cf. (3.57) and (3.67))[33]

$$E_0 = \sqrt{2\sqrt{\lambda}(S+J)}\left[1 + \frac{\frac{3}{8}S + \frac{1}{8}J}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{S^2}{\lambda}\right)\right] . \tag{3.70}$$

In particular, for $S = J$ the leading two terms here become exactly the same as in the energy of the small circular $S = J$ string (3.46) discussed above:[34]

$$E_0 = 2\sqrt{\sqrt{\lambda}S}\left[1 + \frac{S}{2\sqrt{\lambda}} + \mathcal{O}\left(\frac{S^2}{\lambda}\right)\right] . \tag{3.71}$$

The flat-space limits of the two $S = J$ solutions are, however, different — the circular $S = J$ string (3.41) reduces to (3.10) while the folded $S = J$ string still reduced to the folded string rotating in *one* plane (3.62).[35]

In the circular $S = J$ case the leading $\sqrt{S}$ term in the 1-loop correction E_1 happened to cancel out (see (3.53)) and we interpreted this as a cancellation of the 1-loop corrections in (3.19) and in (3.36) if we could formally put them together. If we assume that the leading 1-loop correction in the folded S-string (3.61) and the folded J-string (3.67) energies can

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[32] The SYM operator dual to it may contain terms like $\mathrm{Tr}[\Phi_1, [\Phi_1, \bar{\Phi}_k][\Phi_1, [\Phi_1, \Phi_k]]]$.

[33] This follows from the straightforward combination of the folded string solutions in AdS_3 and in $R \times S^2$ [27]. We thank A. Tirziu for the derivation of this expression.

[34] Let us mention that there is yet another familiar (S, J) string obtained giving the folded string in AdS_5 an angular momentum J in S^5 [33]. In this case the small-spin limit of the classical energy is [31, 33]

$$E_0 = \sqrt{2\sqrt{\lambda}S + J^2}\ \left[1 + \frac{\frac{3}{8}S}{\sqrt{\lambda}} + ...\right] = \sqrt{2\sqrt{\lambda}S}\left[1 + \frac{\frac{3}{8}S + \frac{J^2}{4S}}{\sqrt{\lambda}} + \mathcal{O}(\frac{S^2}{\lambda})\right]$$

One may expect that the corresponding state on the first excited string level should than still have $S = 4$ as in the $J = 0$ case. The corresponding representation $(E, 4, 0; J, 0, 0)$ or $[0, J, 0]_{(2,2)}$ is not, however, in the Konishi multiplet table for $J > 0$ so we will not discuss this case here.

[35] Indeed, the $S = J$ folded string in $AdS_5 \times S^5$in the flat limit is described by $x_1 + ix_2 = a \sin\sigma e^{i\tau}$, $x_3 + ix_4 = a \sin\sigma e^{i\tau}$, so by rotation $x_1' = \frac{x_1+x_3}{\sqrt{2}}$, $x_2' = \frac{x_2+x_4}{\sqrt{2}}$ this is still equivalent to a folded string spinning only in one plane (x_1', x_2') with spin $S' = 2S$.

also be directly superposed (as it is the case for the classical contributions in (3.70)) then the total 1-loop coefficient in the analog of (3.57) would be $a_{01} = -\frac{1}{4} + \frac{1}{4} = 0$, i.e. it would vanish just like in the circular $S = J$ case. Then we would finish with the following result for the 1-loop corrected energy (after shifting $S + J = 2S \to 2S - 2$ to make (3.71) match the flat-space limit)

$$E = 2\sqrt{\sqrt{\lambda}(S-1)}\left[1 + \frac{(S-1)}{2\sqrt{\lambda}} + \mathcal{O}\left(\frac{S^2}{\lambda}\right)\right] + 2 + \mathcal{O}\left(\frac{S}{\sqrt{\lambda}}\right) . \tag{3.72}$$

Here we assumed the same "non-analytic" constant term as in the other two folded string cases (3.65) and (3.68).

Modulo the constant $+2$ shift this happens to be exactly the same expression (3.54) as found earlier in the circular $S = J$ case. Then the choice of $S = J = 2$ gives again a state on the first excited string level. and (3.72) reproduces the same expression for the first two leading terms in (3.55) as in all other previous cases.

The representation corresponding to the folded $S = J = 2$ state is the same as in the circular $S = J = 2$ case, i.e. $(E, 2, 0; 2, 0, 0)$ or $[0, 2, 0]_{(1,1)}$. There are 5 such states in the Konishi table 1 already listed in (3.56); we repeat them again here[36]

$$[0,2,0]_{(1,1)} \;\; : \qquad \Delta_0 = 4 \ (1) \ ; \qquad \Delta_0 = 6 \ (3) \ ; \qquad \Delta_0 = 8 \ (1) \ . \tag{3.73}$$

Given that we identified the circular $S = J = 2$ state with a $\Delta_0 = 4$ state in (3.73), it is natural to assume that the folded $S = J = 2$ state, like the folded $S = 4$ (3.66) and $J = 4$ (3.69) states, should correspond to one of the three $\Delta_0 = 6$ states in representation $[0, 2, 0]_{(1,1)}$ in the Konishi multiplet table.

The proposal is then that the three circular solutions represent Konishi states at level $\Delta_0 = 4$ while the three folded solutions represent Konishi states at level $\Delta_0 = 6$. This appears to be in line with each of these two groups of solutions having distinct flat-space limit (cf. (3.6) and (3.62)).

4 Summary

As we have argued above, the interpolation of semiclassical expressions for 1-loop corrected energies of two classes of spinning string solutions to small values of spins corresponding to quantum string states at the first excited level leads to the following expression (cf. (2.7), (1.9), (1.10))

$$E = 2\sqrt[4]{\lambda} + \Delta_0 - 4 + \frac{1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt[4]{\lambda})^3}\right) . \tag{4.1}$$

Here $\Delta_0 = 4$ for the three states in the Konishi multiplet table $[2, 0, 2]_{(0,0)}$ (3.23), $[0, 0, 0]_{(2,0)}$ (3.40), and $[0, 2, 0]_{(1,1)}$ (3.56) represented by the three circular string configurations, and $\Delta_0 = 6$ for the three states $[0, 0, 0]_{(2,2)}$ (3.66), $[0, 4, 0]_{(0,0)}$ (3.69) and

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[36] As was already mentioned below (3.56), the operator dual to the $\Delta_0 = 4$ state should be the familiar sl(2) sector one $\mathrm{Tr}[\Phi_1(D_{1+i2})^2\Phi_1]$. The operator for the $\Delta_0 = 6$ state may contain terms like $\mathrm{Tr}[\bar{\Phi}_k, D_{1+i2}\Phi_1][\Phi_k, D_{1+i2}\Phi_1]$, etc.

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$[0,2,0]_{(1,1)}$ (3.73) represented by the three folded string configurations. The universality of the coefficients in $E-\Delta_0$ is consistent with the expectation that all gauge-theory states in the same supermultiplet should have the same anomalous dimension. It also lends strong support to the validity of our proposal. Indeed, the $psu(2,2|4)$ generators that could relate the various solutions discussed in this paper are not manifestly realized in the quantum theory based on the GS action. Their realization at the quantum level is highly dependent on the choice of a regularization scheme. The universality of the coefficients in $E-\Delta_0$ found here is a nontrivial confirmation that our methods indeed realize the $psu(2,2|4)$ symmetry algebra at 1-loop level.

In (4.1) we conjectured that 2-loop coefficient b_2 in (1.8) vanishes so that the leading correction to the first three leading terms in strong-coupling expansion is determined by the "analytic" 2-loop term of order $\frac{1}{(\sqrt[4]{\lambda})^3}$.

It is interesting to note that (4.1) has very similar form to the expansion of energy of a massive scalar in AdS_5 with a mass corresponding to the first excited string level (cf. (2.4))

$$E(E-4) = m_0^2 = 4\sqrt{\lambda}\,, \tag{4.2}$$

i.e.

$$E = 2 + \sqrt{4\sqrt{\lambda}+4} = 2\sqrt[4]{\lambda} + 2 + \frac{1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt[4]{\lambda})^3}\right) \,. \tag{4.3}$$

Heuristically, one may argue that the mass of the corresponding 10-d scalar should not receive leading $\alpha' = \frac{1}{\sqrt{\lambda}}$ correction since a candidate for the leading background-dependent correction in the case of the scalar operator (2.3) vanishes for $AdS_5\times S^5$background. The constant 2 in (4.3) would be consistent with (4.1) if the corresponding scalar would be dual to the $\Delta_0=6$ state in the Konishi multiplet. There are indeed three singlet $[0,0,0]_{(0,0)}$ states with $\Delta_0=6$ in the Konishi multiplet table 1, but the significance of this observation remains to be understood.

As follows from our discussion in section 2.2, interpreting E as the solution of the marginality condition for the corresponding string vertex operators, the first two subleading coefficients b_0 and b_1 in (1.5) must be rational because the 1-loop 2-d anomalous dimensions may contain only rational coefficients. At the same time, the semiclassical string computations in [11] and here (3.19) imply that b_3 should already be transcendental, containing $\zeta(3)$. These are robust predictions of our approach.

At the same time, one may wonder if there might be some subtlety in our interpretation of a semiclassical result for the string energy E interpolated to small values of spins as directly representing the quantum string energy.[37] One may suspect that our semiclassical result for $E_0+E_1+...$ (let us denote it $E_{\rm sc}$) computes, in fact, the quantum-corrected string mass $m=m_0+...=\sqrt{\sqrt{\lambda}(n-1)}+...=E_{\rm sc}$. Then to get the value of the quantum string AdS_5 energy one would need still to solve the equation like (4.2), i.e. $E_{\rm q}(E_{\rm q}-4)=E_{\rm sc}^2$. It is easy to see that in this case the value of the coefficient b_1 for a state on the first excited

[37] For example, one may wonder if one may need to shift E by an integer just as we did shift spins to match the flat-space limit.

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level will double from 1 in $E_{\rm sc}$ in (4.1) to 2 in $E_{\rm q}$. To match the right values of b_0 for different states in the Konishi multiplet one will need to use a more complicated ansatz like $E_{\rm q}(E_{\rm q}-4p_1)+p_2=E_{\rm sc}^2$ (with p_1, p_2=const). This prescription, however, seems ad hoc, so we hesitate to advocate it here.

Still, intriguingly, $b_1=2$ appears to be the value coming out of the very recent numerical solution for the strong-coupling expansion of the dimension of the Konishi operator from integrability (Y-system) approach [36].

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A Path integral approach to computation of 1-loop correction to string energy

As discussed at length in, e.g., [7, 34, 35], loop corrections to energy of classical solutions may be efficiently evaluated in the path integral approach in the conformal gauge. The 1-loop correction to the energy of a classical solution (soliton) of the world-sheet theory is given in terms of the logarithm of the determinants of the kinetic operators of the bosonic and fermionic quadratic fluctuations around the solution:

$$E_1=\frac{1}{2\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln\frac{\det K_B}{\det K_F}\quad . \tag{A.1}$$

We are assuming that the solution is stationary in τ (with $t=\kappa\tau$) so that the determinants are 1-dimensional ones. In the closed string case where the theory defined on a spatial cylinder they can be expressed in terms of the characteristic polynomials, $P_B(\omega,n,\mathcal{C})$ and $P_F(\omega,n,\mathcal{C})$, of the bosonic and fermionic fluctuations

$$E_1=\frac{1}{2\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{n=-\infty}^{\infty}\ln\frac{P_B(\omega,n,\mathcal{C})}{P_F(\omega,n,\mathcal{C})}\quad . \tag{A.2}$$

Here $\mathcal{C}$ denotes some charges (rescaled by string tension, $\mathcal{C}=\frac{\mathrm{C}}{\sqrt{\lambda}}$) characterizing the classical solution; κ is also a function of them through the conformal gauge conditions. For each value of n, the ω integral is convergent at large ω (the string sigma-model is UV finite).

The roots of P_B and P_F are the usual characteristic frequencies. If the characteristic polynomials $P_B(\omega, n, \mathcal{C})$ and $P_F(\omega, n, \mathcal{C})$ factorize into products

$$\prod_I [\omega - \omega_I(n, \mathcal{C})][\omega + \omega_I(n, \mathcal{C})] \tag{A.3}$$

then the ω integral may be trivially carried out and one obtains the standard expression for E_1 as a sum over characteristic frequencies ω_I.

The dependence on the charges $\mathcal{C}$ should be extracted from the expressions (A.1) and (A.2) with care. Since the charges $\mathcal{C}$ are parameters of the classical solution, they appear analytically in the quadratic fluctuation Lagrangian and thus in the characteristic equations. The roots of the characteristic equation may, however, depend on fractional powers of $\mathcal{C}$, e.g., on $\sqrt{\mathcal{C}}$. This may occur for a finite set of mode numbers n. We may thus distinguish the two types of contributions: the analytic in $\mathcal{C}$ and the non-analytic in $\mathcal{C}$.

To find the analytic contributions one may consider evaluating the determinants in (A.1) or the ω integral in (A.2) in a perturbative expansion in $\mathcal{C}$. This amounts to interpreting as perturbations all the terms in the string quadratic fluctuation Lagrangian which depend on the parameters of the classical solution. This expansion is thus analogous to the mass insertion formalism in 4d QFT.[38]

The presence of non-analytic $\mathcal{C}$-dependence will manifest itself as a breakdown of this perturbative treatment. In particular, it may happen that at some order in the small $\mathcal{C}$ expansion, the ω integral will be divergent at finite values of ω.[39] By carrying out the expansion of the terms in the summand in equation (A.2) one may be able to identify the mode numbers responsible for potential non-analytic terms.[40] The corresponding fractional power of the charge will lie between the integer powers of $\mathcal{C}$ for which the last convergent and the first divergent ω integrals may occur.

The values of n for which the singularities in the small $\mathcal{C}$ expansion may occur should be analysed separately. While a priory the fractional powers of $\mathcal{C}$ could appear at high orders in the small $\mathcal{C}$ expansion, in all the cases we discussed above they potentially occur as the leading term, even before the first analytic term. Assuming the characteristic equations have the symmetry $(\omega, n) \leftrightarrow (-\omega, -n)$, the leading $\sqrt{\mathcal{C}}$ dependence can then be easily extracted by a simple change of variables in the ω integral. Namely, we are to consider all the apparently singular terms (labelled by n_s) together

$$\ln \prod_{n_s} \frac{P_F(\omega, n_s, \mathcal{C})}{P_B(\omega, n_s, \mathcal{C})} \quad . \tag{A.4}$$

The assumed symmetry of the characteristic equations guarantees that this logarithm is a real function and that the ω integral is well defined.

[38]The GS fermions should be treated with care since their entire kinetic term may be proportional to some charge. In this case one is to redefine the fermions to absorb the leading charge dependence.
[39]The integral over ω is to be convergent at $\omega \to \pm\infty$ due to UV finiteness.
[40]It is possible that additional non-analytic terms may arise from a resummation of the modes.

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For the circular (homogeneous) rotating string solutions discussed in sections 3.1-3.3 the potential non-analytic dependence on the spins $\mathcal{S}$ or $\mathcal{J}$ arises from factors of the type

$$\ln \frac{[(\omega - n_0)^2 - \mathcal{C}]^m}{(\omega - n_0)^{2m}} , \tag{A.5}$$

where m is some even integer.[41] Changing the variable ω, we can then extract the $\sqrt{\mathcal{C}}$ dependence of the 1-loop correction to the energy. The coefficient of the $\sqrt{\mathcal{C}}$ term is given by a well-defined integral which happens to vanish identically.

As an example, let us consider in some detail the case of the small circular string in AdS_5 with $S_1 = S_2$ discussed in section 3.2. As one can check by analyzing the characteristic equations, the only potential non-analytic contributions arise from the modes with numbers $n = \pm 2, \pm 1, 0$.[42] Combining these modes together as

$$\ln \prod_{n=-2}^{2} \frac{P_F(\omega, n, \mathcal{S})}{P_B(\omega, n, \mathcal{S})} \tag{A.6}$$

and using the explicit form of the characteristic polynomials, we find that to extract the leading $\mathcal{S} \to 0$ dependence of the integral of (A.6), the argument of the logarithm in (A.6) can be simplified to[43]

$$\prod_{n=-2}^{2} \frac{P_F(\omega, n, \mathcal{S})}{P_B(\omega, n, \mathcal{S})} \quad \to \quad \frac{[(\omega-1)^2 - \mathcal{S}]^4}{[(\omega-1)^2]^4} \frac{(\omega^2 - \mathcal{S})^8}{(\omega^2)^8} \frac{[(\omega+1)^2 - \mathcal{S}]^4}{[(\omega+1)^2]^2}$$
$$\times \frac{[(\omega-1)^2]^2}{[(\omega-1)^2 - 2\mathcal{S}]^2} \frac{[(\omega+1)^2]^2}{[(\omega+1)^2 - 2\mathcal{S}]^2} . \tag{A.7}$$

Indeed, it is clear that naively expanding (A.6) and (A.7) at small $\mathcal{S}$ leads to singular ω integrals.

Splitting the logarithm of (A.7) into the sum of logarithms of the factors shown above, allows, through simple changes of variables in the ω-integral of each of the resulting terms, to extract the leading $\sqrt{\mathcal{S}}$ dependence as

$$\sqrt{\mathcal{S}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \frac{(\omega^2 - 1)^8}{(\omega^2)^8} \quad . \tag{A.8}$$

The integral here vanishes (as can be seen explicitly by carefully writing the integral as a combination of simple logarithms and shifting the integration variable), implying the vanishing of the coefficient of the leading non-analytic $\sqrt{\mathcal{S}}$ term in the 1-loop correction to the energy.

In general, it would be important to clarify the structure of the small spin expansion and the issue of analytic and non-analytic terms in the 1-loop corrections similar to the one discussed above further, e.g., using other methods of evaluating the 1-loop determinants or attempting to do the summation over modes before expanding in the small-spin parameter.

[41] The power m is even due to the assumed symmetry $(\omega, n) \leftrightarrow (-\omega, -n)$.

[42] One may see this by simply expanding the argument of the logarithm at small $\mathcal{S}$ and noticing the appearance of singularities for finite values of ω.

[43] This essentially amounts to dropping all $\mathcal{S}$-dependence that does not introduce singularities in the ω integral. Such terms necessarily yield only subleading $\mathcal{O}(\mathcal{S})$ contributions.

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Δ_0	$[p_1,q,p_2]_{(s_L,s_R)}=[J_2-J_3,J_1-J_2,J_2+J_3]_{(\frac{S_1+S_2}{2},\frac{S_1-S_2}{2})}$
2	$[0,0,0]_{(0,0)}$
$2+\frac{1}{2}$	$[0,0,1]_{(0,\frac{1}{2})}+[1,0,0]_{(\frac{1}{2},0)}$
$2+1$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})}+[0,0,2]_{(0,0)}+[0,1,0]_{(0,1)+(1,0)}+[1,0,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(0,0)}$
$2+\frac{3}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)}+[0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})}+[1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[2,0,1]_{(0,\frac{1}{2})}$
$2+2$	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)}+[0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})}+[0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})}+[2,0,2]_{(0,0)}+[2,1,0]_{(0,1)}$ $+[0,1,2]_{(1,0)}+[0,2,0]_{2(0,0)+(1,1)}+[1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[1,1,1]_{2(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})}$
$2+\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})+(2,\frac{1}{2})}+[0,0,3]_{(\frac{3}{2},0)}+[0,1,1]_{3(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)}+[0,2,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})}$ $+[1,0,0]_{(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+(\frac{3}{2},1)}+[1,0,2]_{(0,\frac{1}{2})+2(1,\frac{1}{2})}+[1,1,0]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})}$ $+[1,1,2]_{(\frac{1}{2},0)}+[1,2,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[2,0,1]_{(\frac{1}{2},0)+2(\frac{1}{2},1)}+[2,1,1]_{(0,\frac{1}{2})}+[3,0,0]_{(0,\frac{3}{2})}$
$2+3$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}+[0,0,2]_{2(0,0)+(1,0)+2(1,1)+(2,0)}+[0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1)}$ $+[0,1,2]_{2(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})}+[0,2,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})}+[0,2,2]_{(0,0)}+[0,3,0]_{(0,1)+(1,0)}$ $+[1,0,3]_{(1,0)}+[1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)}+[1,2,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{2(0,0)+(0,1)+(0,2)+2(1,1)}$ $+[2,0,2]_{(\frac{1}{2},\frac{1}{2})}+[2,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})}+[2,2,0]_{(0,0)}+[3,0,1]_{(0,1)}+[1,0,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$
$2+\frac{7}{2}$	$[0,0,1]_{2(\frac{1}{2},0)+3(\frac{1}{2},1)+(\frac{3}{2},0)+2(\frac{3}{2},1)+(\frac{3}{2},2)}+[0,0,3]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[0,1,1]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+4(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{1}{2})}$ $+[0,1,3]_{(\frac{1}{2},0)}+[0,2,1]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)}+[0,3,1]_{(0,\frac{1}{2})}+[1,0,0]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+3(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{3}{2})}$ $+[1,0,2]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)}+[1,1,0]_{3(\frac{1}{2},0)+4(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+2(\frac{3}{2},1)}+[1,1,2]_{(0,\frac{1}{2})+(1,\frac{1}{2})}$ $+[1,2,0]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})}+[1,3,0]_{(\frac{1}{2},0)}+[2,0,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})}+[2,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)}$ $+[3,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[3,1,0]_{(0,\frac{1}{2})}$
$2+4$	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)}+[0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}+[0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+2(\frac{3}{2},\frac{3}{2})}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)}+[0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})}+[0,4,0]_{(0,0)}+[1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)}+[1,0,3]_{(\frac{1}{2},\frac{1}{2})}+[0,0,4]_{(0,0)}$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})}+[1,2,1]_{(0,0)+(0,1)+(1,0)}+[2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)}+[2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[2,2,0]_{(\frac{1}{2},\frac{1}{2})}+[3,0,1]_{(\frac{1}{2},\frac{1}{2})}+[4,0,0]_{(0,0)}$
$2+\frac{9}{2}$	$[0,0,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+3(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{3}{2})}+[0,0,3]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[0,1,1]_{3(\frac{1}{2},0)+4(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+2(\frac{3}{2},1)}$ $+[0,1,3]_{(0,\frac{1}{2})}+[0,2,1]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})}+[0,3,1]_{(\frac{1}{2},0)}+[1,0,0]_{2(\frac{1}{2},0)+3(\frac{1}{2},1)+(\frac{3}{2},0)+2(\frac{3}{2},1)+(\frac{3}{2},2)}$ $+[1,0,2]_{2(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})}+[1,1,0]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+4(1,\frac{1}{2})+2(1,\frac{3}{2})+(2,\frac{1}{2})}+[1,1,2]_{(\frac{1}{2},0)+(\frac{1}{2},1)}$ $+[1,2,0]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)}+[1,3,0]_{(0,\frac{1}{2})}+[2,0,1]_{2(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)}+[2,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})}$ $+[3,0,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[3,1,0]_{(\frac{1}{2},0)}$
$2+5$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}+[0,0,2]_{2(0,0)+(0,1)+(0,2)+2(1,1)}+[0,1,0]_{3(0,1)+3(1,0)+2(1,1)+(1,2)+(2,1)}$ $+[0,1,2]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})}+[0,2,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})}+[0,2,2]_{(0,0)}+[0,3,0]_{(0,1)+(1,0)}$ $+[1,0,3]_{(0,1)}+[1,1,1]_{2(0,0)+2(0,1)+2(1,0)+2(1,1)}+[1,2,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{2(0,0)+(1,0)+2(1,1)+(2,0)}$ $+[2,0,2]_{(\frac{1}{2},\frac{1}{2})}+[2,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})}+[2,2,0]_{(0,0)}+[3,0,1]_{(1,0)}+[1,0,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$
$2+\frac{11}{2}$	$[0,0,1]_{(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{1}{2},2)+(\frac{3}{2},0)+(\frac{3}{2},1)}+[0,0,3]_{(0,\frac{3}{2})}+[0,1,1]_{3(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})}+[0,2,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)}$ $+[1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+2(1,\frac{1}{2})+(1,\frac{3}{2})+(2,\frac{1}{2})}+[1,0,2]_{(\frac{1}{2},0)+2(\frac{1}{2},1)}+[1,1,0]_{3(\frac{1}{2},0)+2(\frac{1}{2},1)+(\frac{3}{2},0)+(\frac{3}{2},1)}$ $+[1,1,2]_{(0,\frac{1}{2})}+[1,2,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[2,0,1]_{(0,\frac{1}{2})+2(1,\frac{1}{2})}+[2,1,1]_{(\frac{1}{2},0)}+[3,0,0]_{(\frac{3}{2},0)}$
$2+6$	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)}+[0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})}+[0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})}+[2,0,2]_{(0,0)}+[2,1,0]_{(1,0)}$ $+[0,1,2]_{(0,1)}+[0,2,0]_{2(0,0)+(1,1)}+[1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)}+[1,1,1]_{2(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})}$
$2+\frac{13}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})}+[0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)}+[1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)}+[1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})}+[2,0,1]_{(\frac{1}{2},0)}$
$2+7$	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})}+[0,0,2]_{(0,0)}+[0,1,0]_{(0,1)+(1,0)}+[1,0,1]_{(\frac{1}{2},\frac{1}{2})}+[2,0,0]_{(0,0)}$
$2+\frac{15}{2}$	$[0,0,1]_{(\frac{1}{2},0)}+[1,0,0]_{(0,\frac{1}{2})}$
$2+8$	$[0,0,0]_{(0,0)}$

Table 1. Long Konishi multiplet.

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